Applications II

Agenda

Simulation
- Discrete event simulation
- Carwash simulation
- Call bank simulation

• **Graphs**
- Terminology
- Representation
- Traversal
- Shortest path
- Topological sorting

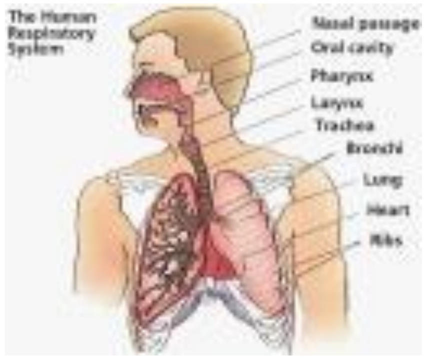
• **Problem complexity**

Simulation

What is simulation?

Experiments with
models
on a computer

Models and systems



The human respiratory system

Model
Representation of a system

System
A chosen extract of reality

Classification of models

- Mental

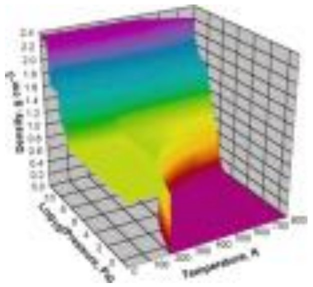
  (e.g., a person’s perception of an object, a “world view”)

- Physical

  (e.g., a model railway, a wax figure, a globe)

- Symbolic

  (e.g., $H_2 + 0 \Rightarrow$ water, $F = ma$)



- **Symbolic models**
  - **Non-mathematical models**
    - **Language models**
      Oral and written descriptions
    - **Graphic models**
      Pictures, drawings
    - **Schematic models**
      Diagrams
  - **Mathematical models**

Mathematical models

- **Static**

  Representation of a system in a given **fixed state**

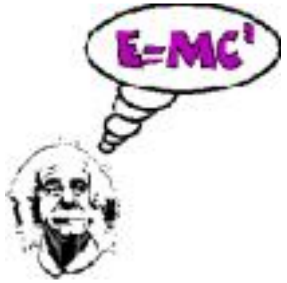
- **Dynamic**

  Representation of a system's behavior over time

Mathematical models

- **Analytica**l

  Relevant questions about the model can be answered by mathematical reasoning (they have a *closed form solution*)



- **Non-analytical**

  Relevant questions about the model are mathematically *unmanageable* (holds for most real-world models)

Simulation
a possible narrowing

Simulation is experimentation with **dynamic**, **non-analytical** models on a computer

Application examples

- **Biology**
  an ecosystem (e.g., the life in a lake), cell growth, the human circulatory system, vegetation)
- **Physics**
  nuclear processes, mechanical movement (e.g., solar systems, launching of rockets)
- **Chemistry**
  chemical reactions, chemical process plants
- **Geography**
  urban development, growth of a population
- **Computer science**
  computers, networks, video games, robotics
- **Management science**
  organizational decision making

Modeling is purposive

Models can neither be false or true.
They can be more or less appropriate in relation to their purpose.

A *good* model is a model that serves its *purpose*.

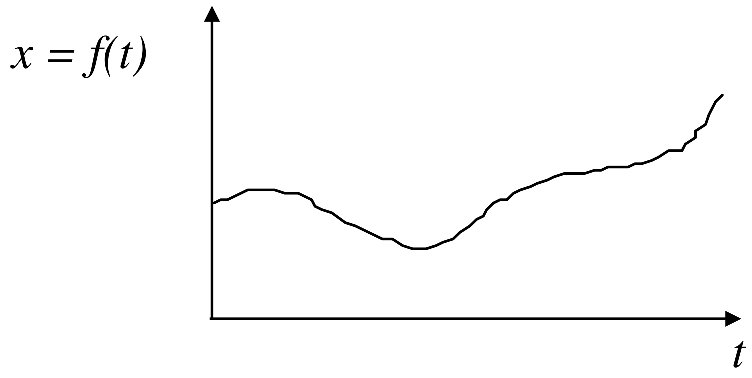
The first step of a modeling process is a clarification of what the model is to be used for.

Abstraction and *aggregation* are used for obtaining *manageable* models.

Abstraction: Ignorance from irrelevant properties
Aggregation: Grouping several things together and considering them as a whole

Dynamic model types

- **Continuous**
  The state of the model is described by variables that vary continuously (without jumps).



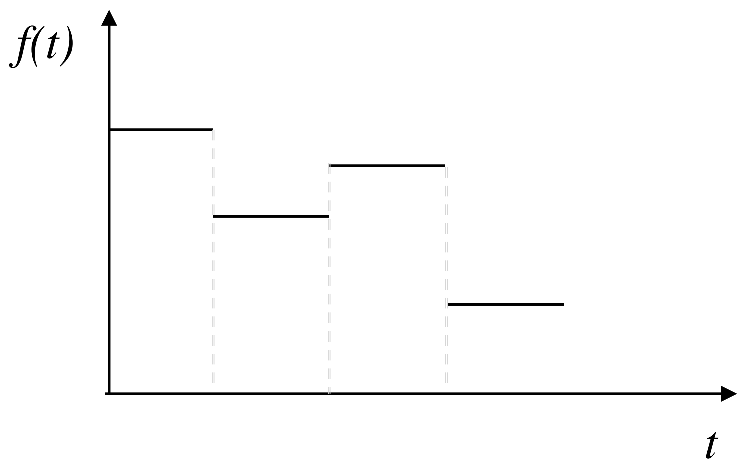
The model is usually expressed as ordinary differential equations and/or difference equations.

$$\boxed{\frac{dx}{dt} = g(x,t)}$$

$$\boxed{x_{next} = x_{now} + g(x_{now}, t)\Delta t}$$

- **Discrete**

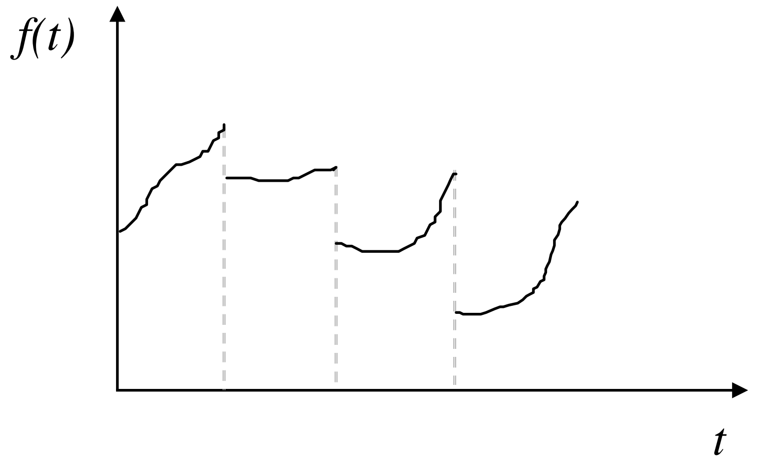
The state of the model is described by variables that vary in jumps (caused by *events*).



Example:
A queue system (customers in a bank, patients in a health centre).

- **Combined continuous and discrete**

  The state may be described by variables that vary continuously and are changed in jumps.



Examples:

Elevator (the movement between floors is continuous, whereas start and stop of the elevator are discrete events).

Refrigerator (the heat exchange with the surroundings is continuous, whereas the thermostat causes discrete events)

Reasons for using simulation

- The system does not exist
- Experiments with the real system are too expensive, too time-consuming, or too dangerous
- Experiments with the real system are practically impossible (e.g., the sun system)

Purpose of simulation

(1) **Decision making**

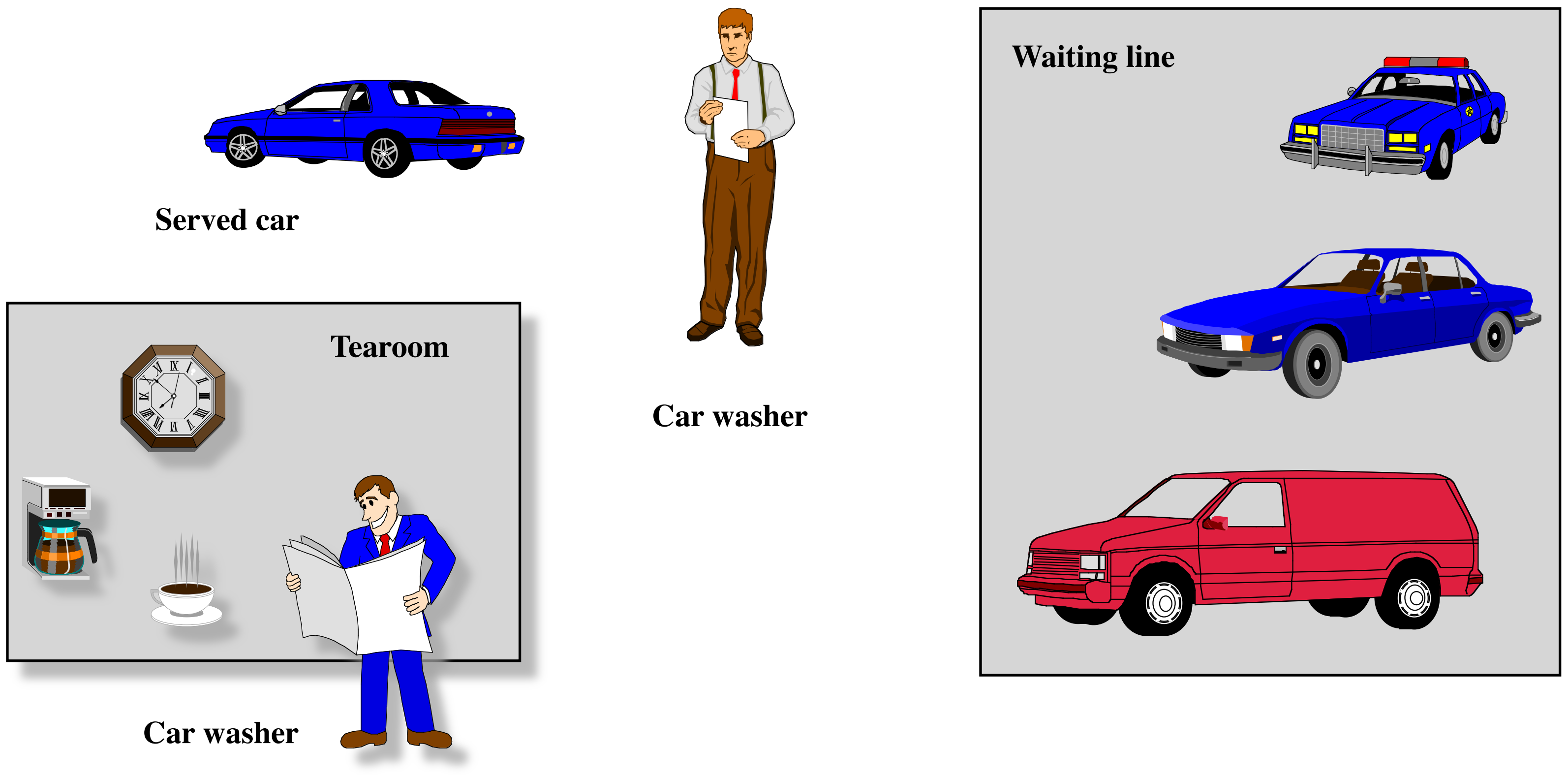
(2) **Insight**

Difficulties of simulation

- May be very expensive, in machine as well as man resources
- Validation is difficult
- Collection of data, and analysis and interpretation of results usually implies good knowledge of statistics

Carwash simulation

Simulation of a carwash



System description

(1) The average time between car arrivals has been estimated at 11 minutes.

(2) When a car arrives, it goes straight into the car wash if this is idle; otherwise, it must wait in a queue.

(3) As long as cars are waiting, the car wash is in continuous operation serving on a first-come basis.

(4) Each service takes exactly 10 minutes.

(5) The car washer starts his day in a tearoom and returns there each time he has no work to do.

(6) The carwash is open 8 hours per day.

(7) All cars that have arrived before the carwash closes down are washed.

Purpose of the simulation

(determines the model)

The purpose is to evaluate how much waiting time is reduced by engaging one more car washer.

Model type

A **discrete event** model

Simulation paradigms

(1) **Event-based** ⟸
(E.g., “A car arrives”, “A wash is finished”)

(2) **Activity-based**
(E.g., “A car is being washed”)

(3) **Process-based** ⟸
(E.g., “A car”, “A car washer”)

Identification of events

(1) A car arrives	(`CarArrival`)
(2) A wash is started	(`StartCarWashing`)
(3) A wash is finished	(`StopCarWashing`)

The package `simulation.event`

by Helsgaun

```
public abstract class Event {
   protected abstract void actions();
   public final void schedule(double evTime);
   public final void cancel();
   public final static double time();
   public final static void runSimulation(double period);
   public final static void stopSimulation();
}
```

Events and their associated actions are defined in subclasses of class `Event`.

```
import simulation.event.*;
import simset.*;
import random.*;
```

```
public class CarWashSimulation extends Simulation {
    int noOfCarWashers, noOfCustomers;
    double openPeriod = 8 * 60, throughTime;
    Head tearoom = new Head(), waitingLine = new Head();
    Random random = new Random(7913);

    CarWashSimulation(int n) { noOfCarWashers = n; ... }

    class CarWasher extends Link {}
    class Car extends Link { double entryTime = time(); }

    class CarArrival extends Event {...}
    class StartCarWashing extends Event {...}
    class StopCarWashing extends Event {...}

    public static void main(String args[]) {
        new CarWashSimulation(2);
    }
}
```

The constructor in `CarWashSimulation`

```
CarWashSimulation(int n) {
    noOfCarWashers = n;
    for (int i = 1; i <= noOfCarWashers; i++)
        new CarWasher().into(tearoom);
    new CarArrival().schedule(0);
    runSimulation(openPeriod + 1000000);
    report();
}
```

CarArrival

```
class CarArrival extends Event {
    public void actions() {
        if (time() <= openPeriod) {
            new Car().into(waitingLine);
            if (!tearoom.empty())
                new StartCarWashing().schedule(time());
            new CarArrival().schedule(time() +
                                      random.negexp(1 / 11.0));
        }
    }
}
```

StartCarWashing

```
class StartCarWashing extends Event {
    public void actions() {
        CarWasher theCarWasher = (CarWasher) tearoom.first();
        theCarWasher.out();
        Car theCar = (Car) waitingLine.first();
        theCar.out();
        new StopCarWashing(theCarWasher, theCar).
            schedule(time() + 10);
    }
}
```

StopCarWashing

```
class StopCarWashing extends Event {
    CarWasher theCarWasher;
    Car theCar;

    StopCarWashing(CarWasher washer, Car car) {
        theCarWasher = washer; theCar = car;
    }

    public void actions() {
        theCarWasher.into(tearoom);
        if (!waitingLine.empty())
            new StartCarWashing().schedule(time());
        noOfCustomers++;
        throughTime += time() - theCar.entryTime;
    }
}
```

The method **`report`**

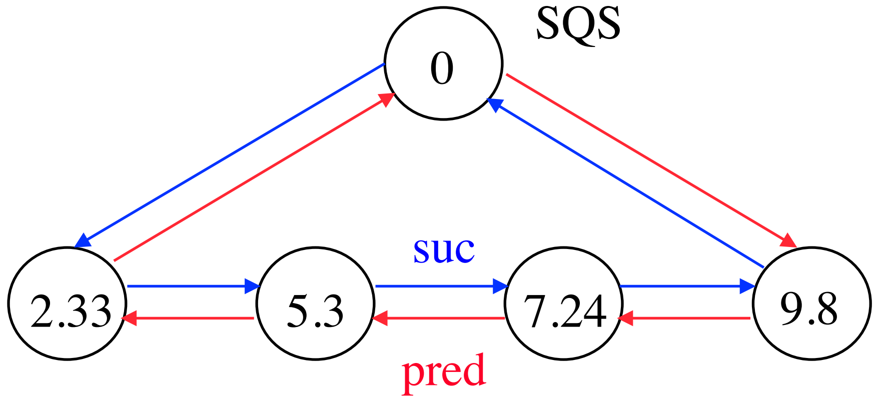
```
void report() {
    System.out.println(noOfCarWashers +
                       " car washer simulation");
    System.out.println("No.of cars through the system = " +
                       noOfCustomers);
    System.out.printf("Av.elapsed time = %1.2f\n",
                      throughTime / noOfCustomers);
}
```

Experimental results

```
1 car washer simulation
No.of cars through the system = 43
Av.elapsed time = 29.50

2 car washer simulation
No.of cars through the system = 43
Av.elapsed time = 12.46

3 car washer simulation
No.of cars through the system = 43
Av.elapsed time = 10.51
```

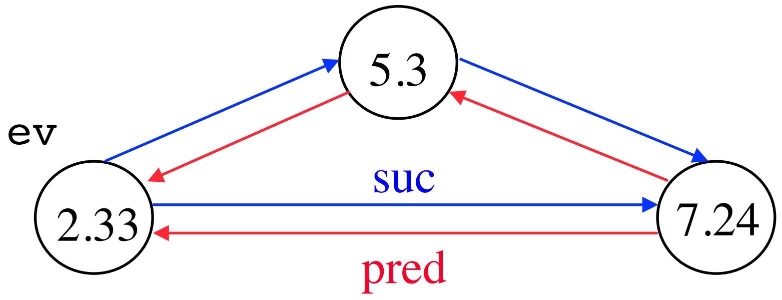
Implementation of the package `simulation.event`

Scheduled events are kept in a circular two-way list, SQS, sorted in increasing order of their associated event times.

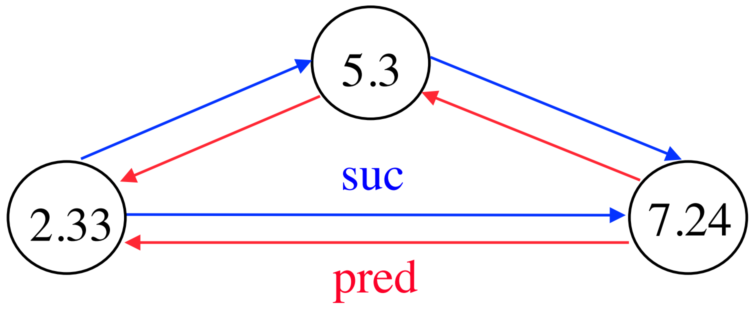


```
public abstract class Event {
    protected abstract void actions();
    ...
    private final static Event SQS = new Event() {
        { pred = suc = this; }
        protected void actions() {}
    };

    private static double time = 0;

    private double eventTime;
    private Event pred, suc;
}
```

```
public void schedule(final double evTime) {
    if (evTime < time)
        throw new RuntimeException
              ("attempt to schedule event in the past");
    cancel();
    eventTime = evTime;
    Event ev = SQS.pred;
    while (ev.eventTime > eventTime)
        ev = ev.pred;
    pred = ev;
    suc = ev.suc;
    ev.suc = suc.pred = this;
}
```



```
public void cancel() {
    if (suc != null) {
        suc.pred = pred;
        pred.suc = suc;
        suc = pred = null;
    }
}
```

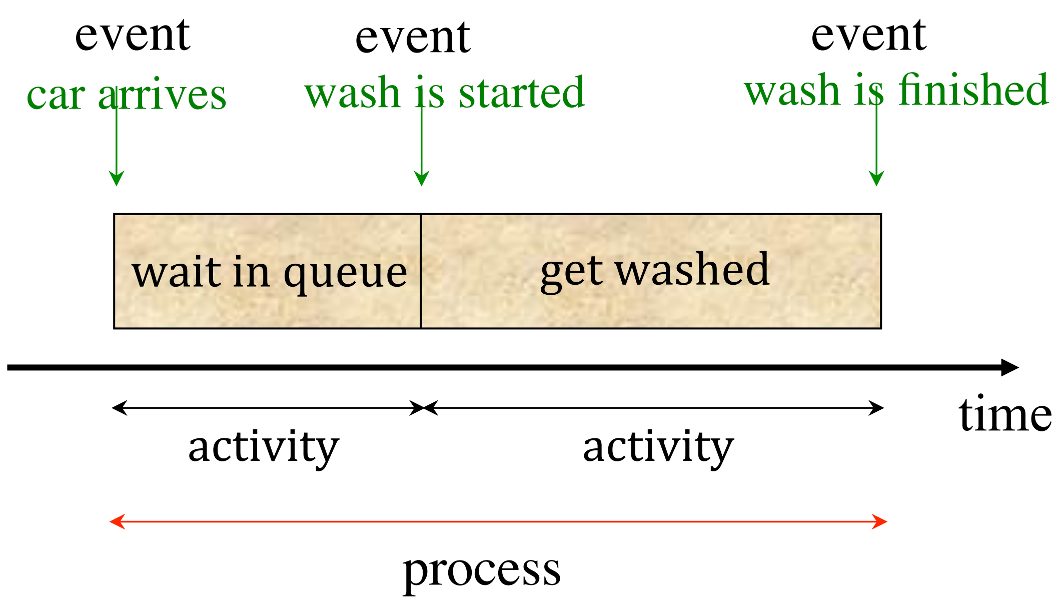


```
public static void runSimulation(double period) {
    time = 0;
    while (SQS.suc != SQS) {
        Event ev = SQS.suc;
        time = ev.eventTime;
        if (time > period)
            break;
        ev.cancel();
        ev.actions();
    }
    stopSimulation();
}
```

```
public static void stopSimulation() {
    while (SQS.suc != SQS)
        SQS.suc.cancel();
}
```

Process-based simulation

A **process** is a system component that executes a sequence of activities in simulated time.



Identification of processes

(1) `Car`

(2) `CarWasher`

(3) `CarGenerator`

The package `javaSimulation`

by Keld Helsgaun

```
public abstract class Process extends Link {
   protected abstract void actions();

   public static double time();
   public static void activate(Process p);
   public static void hold(double t);
   public static void passivate();
   public static void wait(Head q);
}
```

Processes and their associated actions are defined in subclasses of class `Process`.

```
import javaSimulation.*;
import javaSimulation.Process;
```

```
public class CarWashSimulation extends Process {
    int noOfCarWashers, noOfCustomers;
    double openPeriod = 8 * 60, throughTime;
    Head tearoom = new Head(), waitingLine = new Head();
    Random random = new Random(7913);

    CarWashSimulation(int n) { noOfCarWashers = n; }

    public void actions() {...}

    class Car extends Process {...}
    class CarWasher extends Process {...}
    class CarGenerator extends Process {...}

    public static void main(String args[]) {
        activate(new CarWashSimulation(2));
    }
}
```

The actions of the main process

```
public void actions() {
    for (int i = 1; i <= noOfCarWashers; i++)
        new CarWasher().into(tearoom);
    activate(new CarGenerator());
    hold(openPeriod + 1000000);
    report();
}
```

Class `CarGenerator`

```
class CarGenerator extends Process {
    public void actions() {
        while (time() <= openPeriod) {
            activate(new Car());
            hold(random.negexp(1 / 11.0));
        }
    }
}
```

Class **`Car`**

```
class Car extends Process {
    public void actions() {
        double entryTime = time();
        into(waitingLine);
        if (!tearoom.empty())
            activate((CarWasher) tearoom.first());
        passivate();
        noOfCustomers++;
        throughTime += time() - entryTime;
    }
}
```

Class **CarWasher**

```
class CarWasher extends Process {
    public void actions() {
        while (true) {
            out();
            while (!waitingLine.empty()) {
                Car served = (Car) waitingLine.first();
                served.out();
                hold(10);
                activate(served);
            }
            wait(tearoom);
        }
    }
}
```

A call bank simulation

A call bank consists of a large number of operators who handle phone calls. An operator is reached by dialing one phone number.

If any of the operators are available, the user is connected to one of them.

If all operators are already taking a phone, the phone will give a busy signal, and the user will hang up.

Simulate the service provided by the pool of operators. The variables are

- The number of operators in the bank
- The probability distribution that governs dial-in attempts
- The probability distribution that governs connect time
- The length of time the simulation is to be run

Sample output

```
User 0 dials in at time 0 and connects for 1 minute
User 0 hangs up at time 1
User 1 dials in at time 1 and connects for 5 minutes
User 2 dials in at time 2 and connects for 4 minutes
User 3 dials in at time 3 and connects for 11 minutes
User 4 dials in at time 4 but gets busy signal
User 5 dials in at time 5 but gets busy signal
User 6 dials in at time 6 but gets busy signal
User 1 hangs up at time 6
User 2 hangs up at time 6
User 7 dials in at time 7 and connects for 8 minutes
User 8 dials in at time 8 and connects for 6 minutes
User 9 dials in at time 9 but gets busy signal
User 10 dials in at time 10 but gets busy signal
User 11 dials in at time 11 but gets busy signal
User 12 dials in at time 12 but gets busy signal
User 13 dials in at time 13 but gets busy signal
User 3 hangs up at time 14
User 14 dials in at time 14 and connects for 6 minutes
User 8 hangs up at time 14
User 15 dials in at time 15 and connects for 3 minutes
User 7 hangs up at time 15
User 16 dials in at time 16 and connects for 5 minutes
User 17 dials in at time 17 but gets busy signal
User 15 hangs up at time 18
User 18 dials in at time 18 and connects for 7 minutes
```

figure 13.4

Sample output for the modem bank simulation involving three modems: A dial-in is attempted every minute; the average connect time is 5 minutes; and the simulation is run for 18 minutes

figure 13.5

The Event class used for modem simulation

```
/**
 * The event class.
 * Implements the Comparable interface
 * to arrange events by time of occurrence.
 * (nested in ModemSim)
 */
private static class Event implements Comparable<Event>
{
    static final int DIAL_IN = 1;
    static final int HANG_UP = 2;

    public Event( )
    {
        this( 0, 0, DIAL_IN );
    }

    public Event( int name, int tm, int type )
    {
        who  = name;
        time = tm;
        what = type;
    }

    public int compareTo( Event rhs )
    {
        return time - rhs.time;
    }

    int who;        // the number of the user
    int time;       // when the event will occur
    int what;       // DIAL_IN or HANG_UP
}
```

```java
import java.util.Random;
import java.util.PriorityQueue;

// ModemSim clas interface: run a simulation
//
// CONSTRUCTION: with three parameters: the number of
//     modems, the average connect time, and the
//     interarrival time
//
// ******************PUBLIC OPERATIONS*********************
// void runSim( )       --> Run a simulation

public class ModemSim
{
    public ModemSim( int modems, double avgLen, int callIntrvl )
      { /* Figure 13.7 */ }

      // Run the simulation.
    public void runSim( long stoppingTime )
      { /* Figure 13.9 */ }

      // Add a call to eventSet at the current time,
      // and schedule one for delta in the future.
    private void nextCall( int delta )
      { /* Figure 13.8 */ }

    private Random r;                       // A random source
    private PriorityQueue<Event> eventSet;  // Pending events

        // Basic parameters of the simulation
    private int freeModems;          // Number of modems unused
    private double avgCallLen;       // Length of a call
    private int freqOfCalls;         // Interval between calls

    private static class Event implements Comparable<Event>
      { /* Figure 13.5 */ }
}
```

figure 13.6

The ModemSim class skeleton

figure 13.7

The `ModemSim` constructor

```
/**
 * Constructor.
 * @param modem number of modems.
 * @param avgLen averge length of a call.
 * @param callIntrvl the average time between calls.
 */
public ModemSim( int modems, double avgLen, int callIntrvl )
{
    eventSet    = new PriorityQueue<Event>( );
    freeModems  = modems;
    avgCallLen  = avgLen;
    freqOfCalls = callIntrvl;
    r           = new Random( );
    nextCall( freqOfCalls );  // Schedule first call
}
```

figure 13.8

The `nextCall` method places a new `DIAL_IN` event in the event queue and advances the time when the next `DIAL_IN` event will occur

```
    private int userNum = 0;
    private int nextCallTime = 0;

    /**
     * Place a new DIAL_IN event into the event queue.
     * Then advance the time when next DIAL_IN event will occur.
     * In practice, we would use a random number to set the time.
     */
    private void nextCall( int delta )
    {
        Event ev = new Event( userNum++, nextCallTime, Event.DIAL_IN );
        eventSet.insert( ev );
        nextCallTime += delta;
    }
```

```java
/**
 * Run the simulation until stoppingTime occurs.
 * Print output as in Figure 13.4.
 */
public void runSim( long stoppingTime )
{
    Event e = null;
    int howLong;

    while( !eventSet.isEmpty( ) )
    {
        e = eventSet.remove( );

        if( e.time > stoppingTime )
            break;

        if( e.what == Event.HANG_UP )    // HANG_UP
        {
            freeModems++;
            System.out.println( "User " + e.who +
                                " hangs up at time " + e.time );
        }
        else                          // DIAL_IN
        {
            System.out.print(  "User " + e.who +
                               " dials in at time " + e.time + " " );
            if( freeModems > 0 )
            {
                freeModems--;
                howLong = r.nextPoisson( avgCallLen );
                System.out.println(  "and connects for "
                                   + howLong + " minutes" );
                e.time += howLong;
                e.what = Event.HANG_UP;
                eventSet.add( e );
            }
            else
                System.out.println( "but gets busy signal" );

            nextCall( freqOfCalls );
        }
    }
}
```

figure 13.9

The basic simulation routine

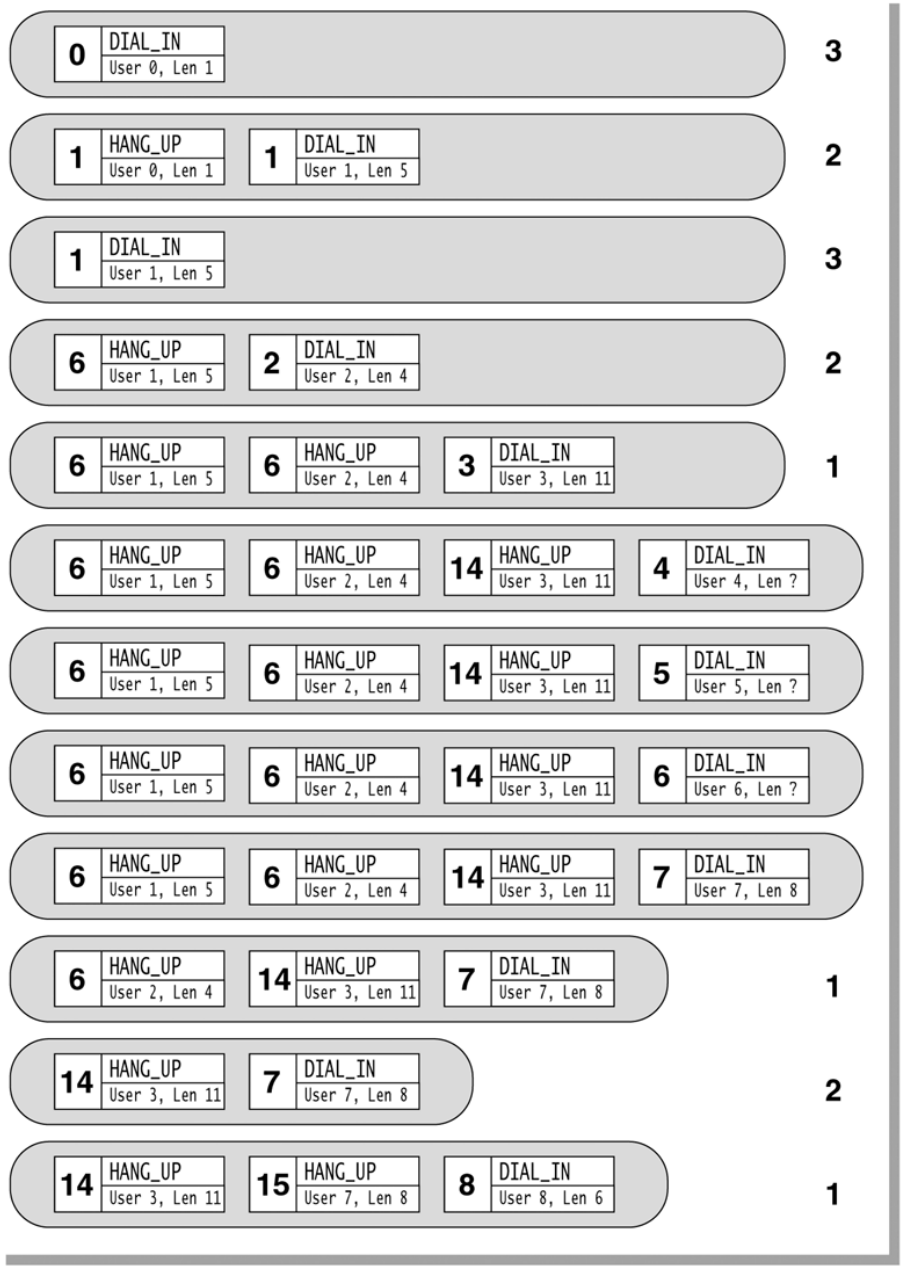


figure 13.10

The priority queue for modem bank simulation after each step

The time at which each event occurs is shown in boldface.

The number of free operators (if any) are shown to the right of the priority queue.

figure 13.11

A simple `main` to test the simulation

```
/**
 * Quickie main for testing purposes.
 */
public static void main( String [ ] args )
{
    ModemSim s = new ModemSim( 3, 5.0, 1 );
    s.runSim( 20 );
}
```

Using `simulation.event`

```
public class CallSim extends Simulation {
    public CallSim(int operators, double avgLen,
                   int callIntrvl) {
        availableOperators = operators;
        avgCallLen = avgLen;
        freqOfCalls = callIntrvl;
    }

    class DialIn extends Event { ... }
    class HangUp extends Event { ... }

    public static void main(String[] args) {
        new CallSim(3, 5.0, 1);
        new DialIn(0).schedule(0.0);
        runSimulation(20);
    }

    int availableOperators, freqOfCalls;
    double avgCallLen;
    Random r = new Random();
}
```

```java
class DialIn extends Event {
    DialIn(int who) { this.who = who; }

    @Override public void actions() {
        System.out.print("User " + who +
                         " dials in at time " + time() + " ");
        if (availableOperators > 0) {
            availableOperators--;
            int howLong = r.poisson(avgCallLen);
            System.out.println("and connects for " +
                               howLong + " minutes");
            new HangUp(who).schedule(time() + howLong);
        } else
            System.out.println("but gets busy signal");
        new DialIn(who + 1).schedule(time() + freqOfCalls);
    }

    int who;
}
```

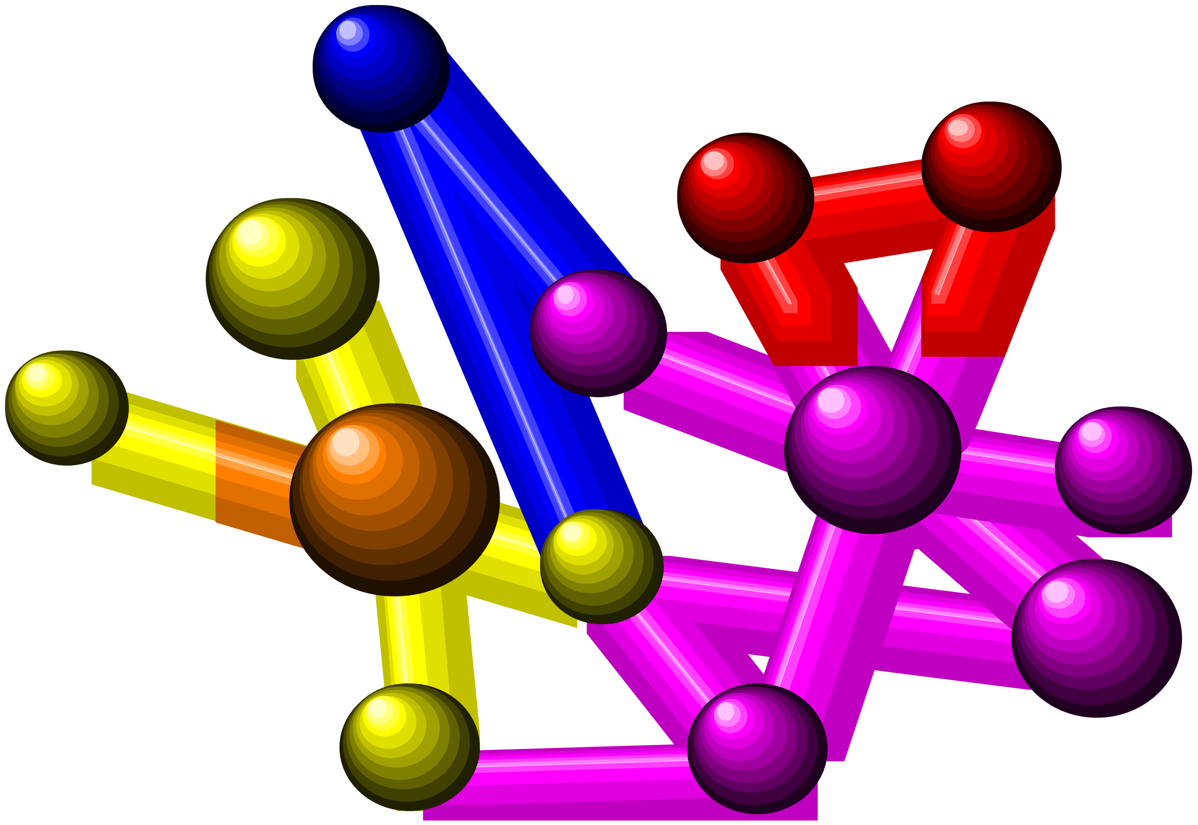
```
class HangUp extends Event {
    HangUp(int who) { this.who = who; }

    @Override public void actions() {
        availableOperators++;
        System.out.println("User " + who +
                           " hangs up at time " + time());
    }

    int who;
}
```

Using `javaSimulation`

```
public class CallSim extends Process {
    public CallSim(int operators, double avgLen,
                   int callIntrvl, int stopTime) {
        availableOperators = operators; avgCallLen = avgLen;
        freqOfCalls = callIntrvl; simTime = stopTime;
    }

    @Override public void actions() {
        activate(new User(0));
        hold(simTime);
    }

    class User extends Process { ... }

    public static void main(String[] args) {
        activate(new CallSim(3, 5.0, 1, 20));
    }

    int availableOperators, freqOfCalls, simTime;
    double avgCallLen;
    Random r = new Random();
}
```

```java
class User extends Process {
    User(int who) { this.who = who; }

    @Override public void actions() {
        activate(new User(who + 1), delay, freqOfCalls);
        System.out.print("User " + who +
                         " dials in at time " + time() + " ");
        if (availableOperators > 0) {
            availableOperators--;
            int howLong = r.poisson(avgCallLen);
            System.out.println("and connects for " +
                               howLong + " minutes");
            hold(howLong);
            availableOperators++;
            System.out.println("User " + who +
                               " hangs up at time " + time());
        } else
            System.out.println("but gets busy signal");
    }

    int who;
}
```

Graphs

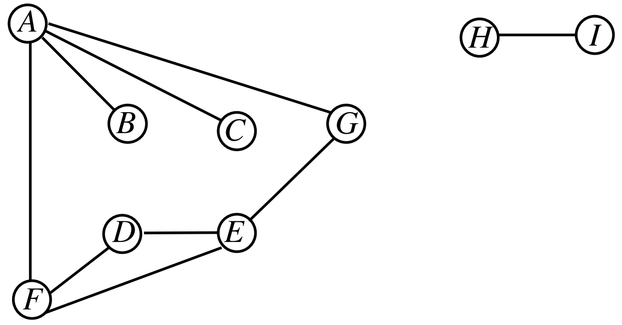


Graphs

A **graph** is a useful abstract concept.

Intuitive definition: A graph is a set of *objects* and a set of *relations* between these objects.

Mathematical definition: A graph $G = (V, E)$ is a finite set of *vertices*, V, (or *nodes*) and a finite set of *edges*, E, where each edge connects two vertices ($E \subseteq V \times V$) .



$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{(A,B), (A,C), (A,F), (A,G), (D,E), (D,F), (E,F), (E,G), (H,I)\}$

Applications

Anything involving relationships among objects can be modeled as a graph

Traffic networks:
Vertices: cities, crossroads
Edges: roads

Electric circuits:
Vertices: devices
Edges: wires

Organic molecules:
Vertices: atoms
Edges: bonds

Game graphs:
Vertices: board positions
Edges: moves

Applications
(continued)

Software systems:

Vertices: methods
Edges: method *A* calls method *B*

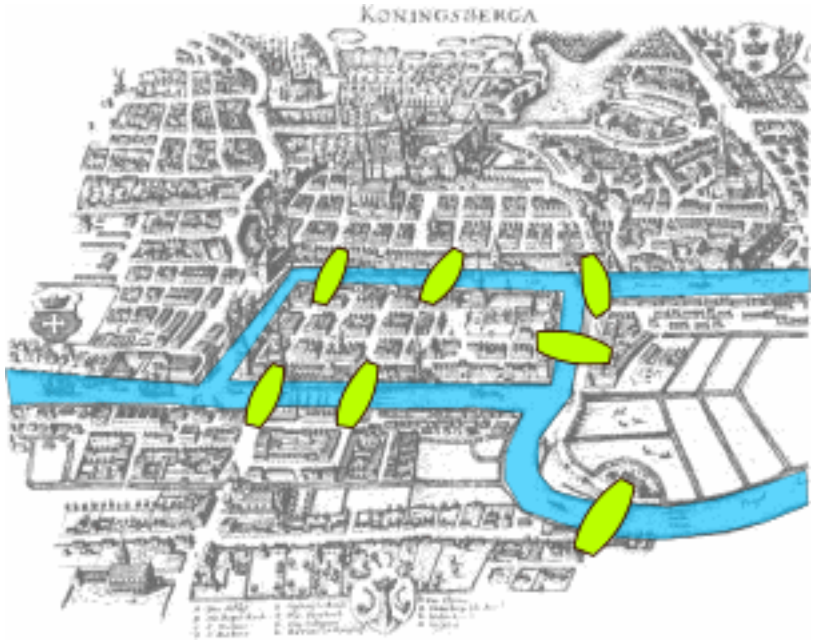
Object-oriented design (UML diagramming):

Vertices: classes/objects
Edges: inheritance, aggregation, association

Project planning:

Vertices: subtasks
Edges: dependencies (subtask *A* must finish be before subtask *B* can start)

Historical foundation of graph theory

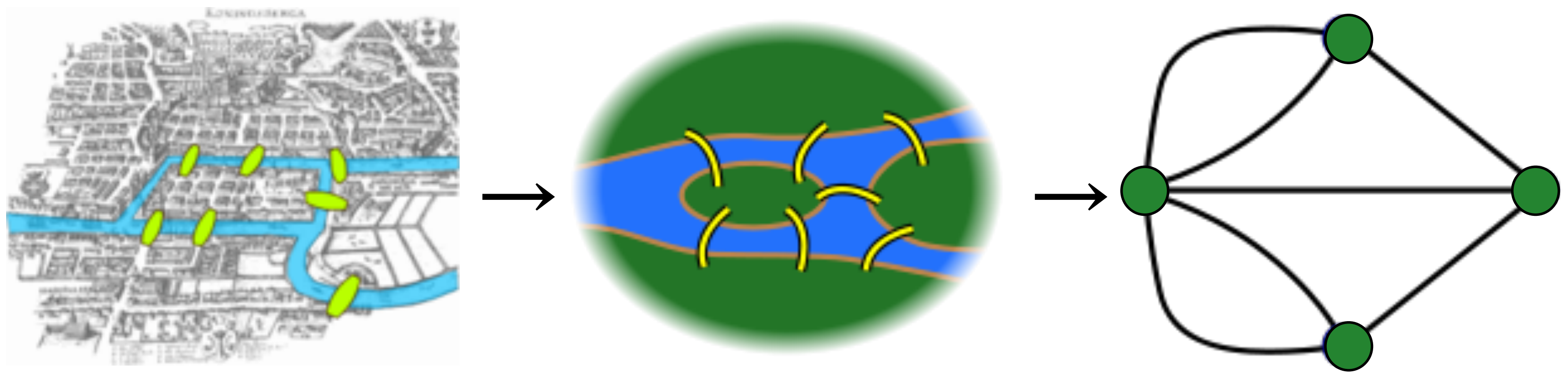


Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

The problem was to find a walk through the city that would cross each bridge once and only once. Euler proved in 1735 that this problem has no solution.

Euler’s analysis

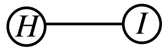
L. Euler, 1707-83



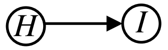
During any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it.
Now if every bridge is traversed exactly once it follows that for each land mass (except possibly for the ones chosen for the start and finish), the number of bridges touching that land mass is **even** (half of them, in the particular traversal, will be traversed "toward" the landmass, the other half "away" from it).
However, all the four land masses are touched by an **odd** number of bridges.

Terminology

The two vertices of an edge is called its **end vertices**.



If an edge is a ordered pair of end vertices, then the edge is said to be **directed**. This is indicated on the visual representation by drawing the edge as an arrow.



A **directed graph** (or **digraph)** is a graph in which all edges are directed.

A **undirected graph** is a graph in which no edges are directed.

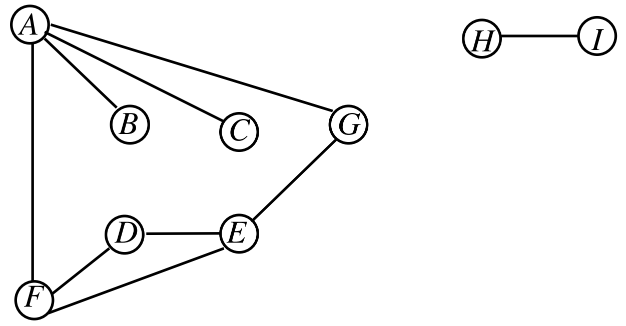
Terminology

(continued)

A **path** is a sequence of vertices connected by edges.

A **simple path** is a path in which all vertices are distinct.

A **cycle** is a path that is simple, except that the first and last vertex are the same.



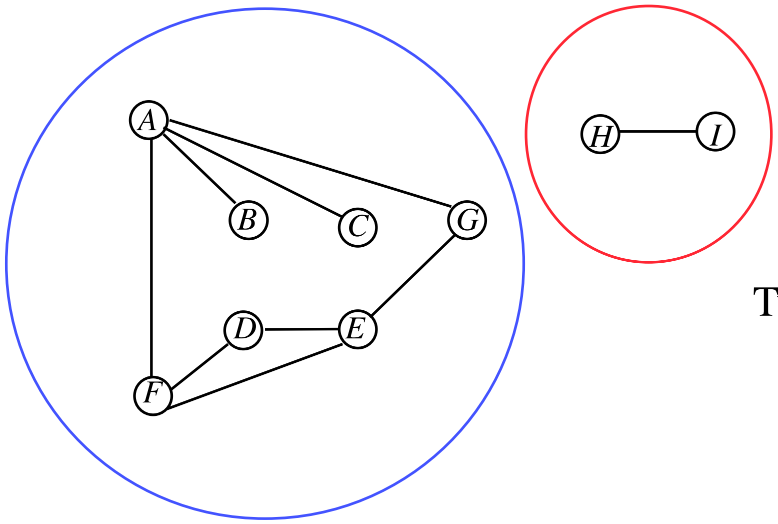
Cycles: *FDEF*, *AFEGA*, and *AFDEGA*

Terminology
(continued)

A graph $G' = (V', E')$ is a **subgraph** of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

A graph is said to be **connected** if, for every two vertices *u* and *v*, there is a path from *u* to *v* or a path from *v* to *u*.

A graph, which is not strongly connected, consists of two or more connected subgraphs, called **components**.



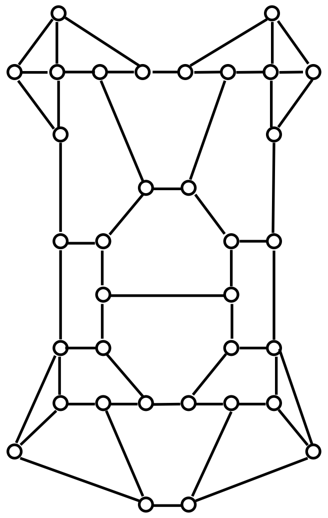
Two components

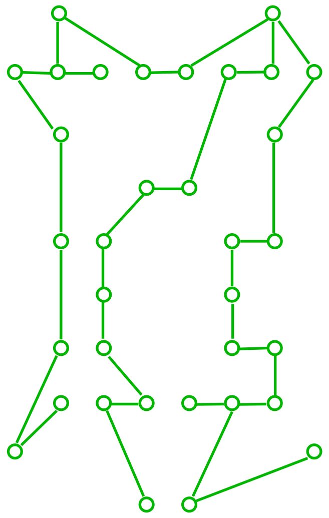
Terminology
(continued)

A **tree** is a connected graph without cycles.

A **forest** is a set of disjoint trees.

A **spanning tree** for a graph *G* is a tree composed of all vertices of *G* and some (or perhaps all) of its edges.


Graf *G*


Spanning tree for *G*

Terminology
(continued)

A graph in which every pair of vertices are connected by a unique edge is said to be **complete**.

[for an undirected complete graph: $|E| = |V|(|V|-1)/2$]

A **dense** graph is a graph in which the number of edges is close to the maximal number of edges.

A **sparse** graph is a graph with only a few edges.

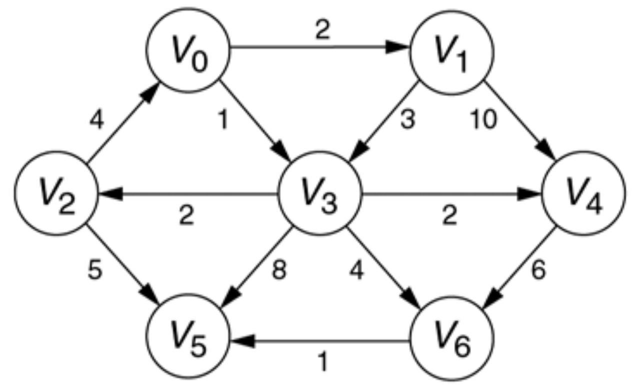
A graph is a **weighted graph** if a number (weight) is assigned to each edge.

[weights usually represent costs]

A directed weighted graph

figure 14.1

A directed graph



Basic graph problems

Paths:

Is there a path from *A* to *B*?

Cycles:

Does the graph contain a cycle?

Connectivity (spanning tree):

Is there a way to connect all vertices?

Biconnectivity:

Will the graph become disconnected if one vertex is removed?

Planarity:

Is there a way to draw the graph without edges crossing?

Basic graph problems

(continued)

Shortest path:

What is the shortest way from *A* to *B*?

Longest path:

What is the longest way from *A* to *B*?

Minimal spanning tree:

What is the cheapest way to connect all vertices?

Hamiltonian cycle:

Is there a way to visit all the vertices without visiting the same vertex twice?

Traveling salesman problem:

What is the shortest Hamiltonian cycle?

Representation of graphs

Graphs are *abstract* mathematical objects.
Algorithms have to work with *concrete* representations.

Many different representations are possible. The choice is decided by algorithms and graph types (sparse/dense, weighted/unweighted, directed/undirected).

Three data structures will be described:

(1) **edge set**
(2) **adjacency matrix**
(3) **adjacency lists**

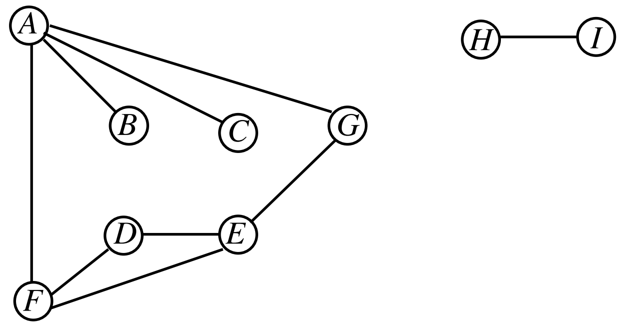
(1) Edge set

```
class Graph {
    Set<Edge> edges;
}
```

```
class Edge {
    Vertex source, dest;
    double cost;
}
```

```
class Vertex {
    String name;
}
```

(2) Adjacency matrix

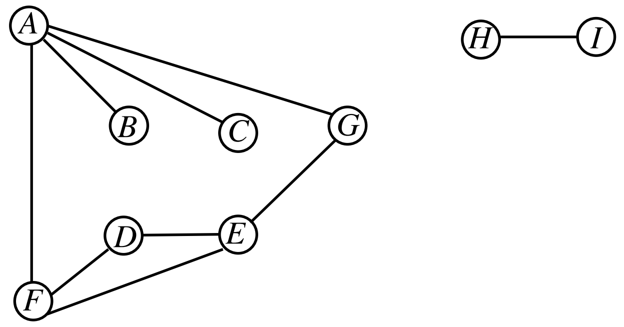


	A	B	C	D	E	F	G	H	I
A	0	1	1	0	0	1	1	0	0
B	1	0	0	0	0	0	0	0	0
C	1	0	0	0	0	0	0	0	0
D	0	0	0	0	1	1	0	0	0
E	0	0	0	1	0	1	1	0	0
F	1	0	0	1	1	0	0	0	0
G	1	0	0	0	1	0	0	0	0
H	0	0	0	0	0	0	0	0	1
I	0	0	0	0	0	0	0	1	0

```
class Graph { // unweighted
    boolean[][] adjMatrix;
}
```

```
class Graph { // weighted
    double[][] adjMatrix;
}
```

(3) Adjacency lists



$$\begin{tikzcd}
A: & F \arrow[r] & C \arrow[r] & B \arrow[r] & G \arrow[r] & \square \\
B: & A \arrow[r] & \square \\
C: & A \arrow[r] & \square \\
D: & F \arrow[r] & E \arrow[r] & \square \\
E: & G \arrow[r] & F \arrow[r] & D \arrow[r] & \square \\
F: & A \arrow[r] & E \arrow[r] & D \arrow[r] & \square \\
G: & E \arrow[r] & A \arrow[r] & \square \\
H: & I \arrow[r] & \square \\
I: & H \arrow[r] & \square
\end{tikzcd}$$

(3) Adjacency lists

```
class Graph {
    Map<String,Vertex> vertexMap;
}
```

```
class Vertex {
    String name;           // Vertex name
    List<Edge> adj;        // Adjacent vertices
}
```

```
class Edge {
    Vertex dest;           // Second vertex of edge
    double cost;           // Edge weight
}
```

figure 14.1

A directed graph

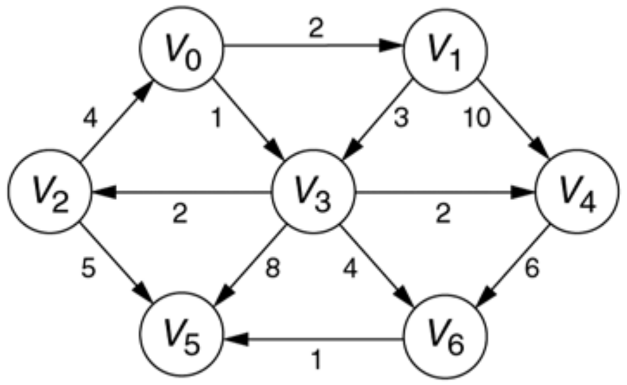
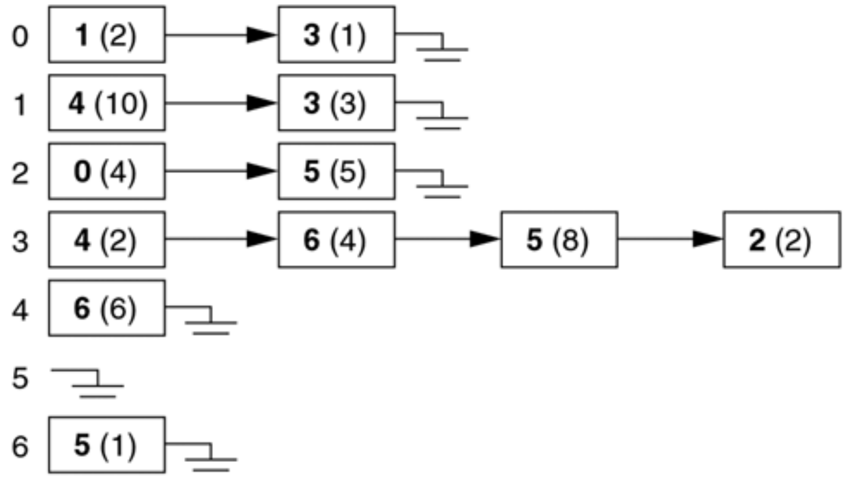


figure 14.2

Adjacency list representation of the graph shown in Figure 14.1; the nodes in list i represent vertices adjacent to i and the cost of the connecting edge.



Comparison of representations

Space requirements:

Edge set:	$O(\|E\|)$
Adjacency matrix:	$O(\|V\|^2)$
Adjacency lists:	$O(\|V\| + \|E\|)$

Choice of representation affects algorithm efficiency

Time complexity (worst case):

Is there an edge from A to B?

Edge set:	$O(\|E\|)$
Adjacency matrix:	$O(1)$
Adjacency lists:	$O(\|V\|)$

Is there an edge from A to anywhere?

Edge set:	$O(\|E\|)$
Adjacency matrix:	$O(\|V\|)$
Adjacency lists:	$O(1)$

Traversing graphs

Goal: "visit" every vertex of the graph.

Depth-first traversal (recursive):

```
*  Mark all vertices as "unvisited"
*  Visit vertex 1
*  To visit a vertex v:
        * mark it
        * (recursively) visit all unmarked vertices
          connected to v by an edge
```

Solves some simple graph problems:
connectivity, cycles

Basis for solving difficult graph problems:
biconnectivity, planarity

Implementation of depth-first traversal

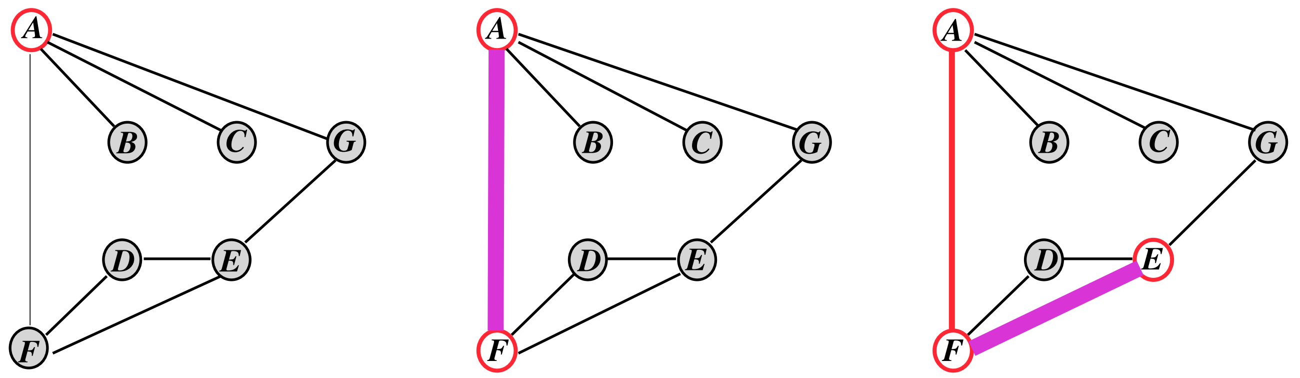
(adjacency lists)

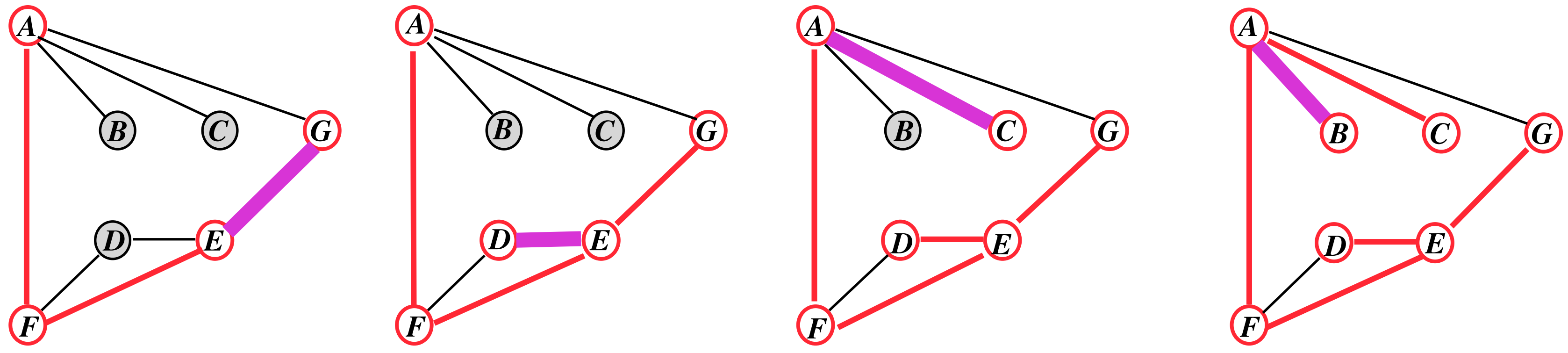
```
class Vertex {
    String name;
    List<Edge> adj;
    boolean visited;

    void visit() {
        visited = true;
        for (Edge e : adj) {
            Vertex w = e.dest;
            if (!w.visited)
                w.visit();
        }
    }
}
```

Time complexity: $O(|E|)$

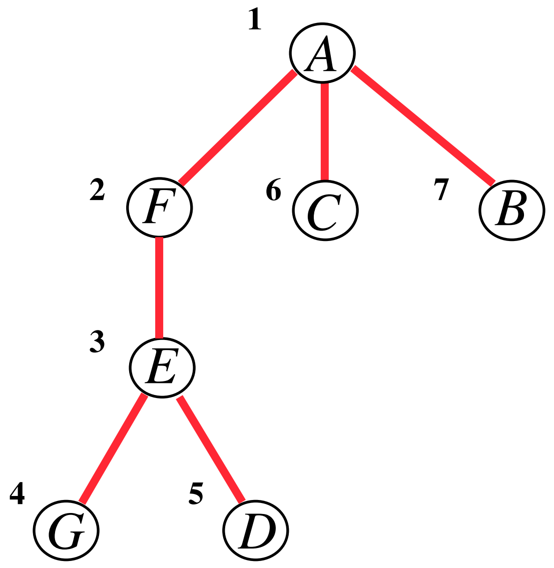
Depth-first traversal of a component

A: F C B G
B: A
C: A
D: F E
E: G F D
F: A E D
G: E A





Depth-first traversal of a component results in a depth-first tree



A depth-first traversal of a connected graph represented by adjacency lists requires $O(|E|)$ time

Non-recursive depth-first traversal

Use an explicit stack of vertices.

```
void traverse(Vertex startVertex) {
    Stack<Vertex> stack = new Stack<Vertex>();
    stack.push(startVertex);
    startVertex.visited = true;
    while (!stack.empty()) {
        Vertex v = stack.pop();
        for (Edge e : v.adj) {
            Vertex w = e.dest;
            if (!w.visited) {
                stack.push(w);
                w.visited = true;
            }
        }
    }
}
```

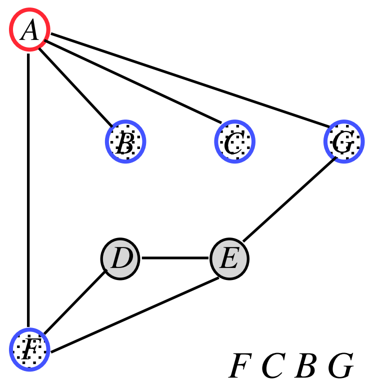
Breadth-first traversal

If the stack is replaced by a queue, the graph will be traversed in *breadth-first order* (level order).

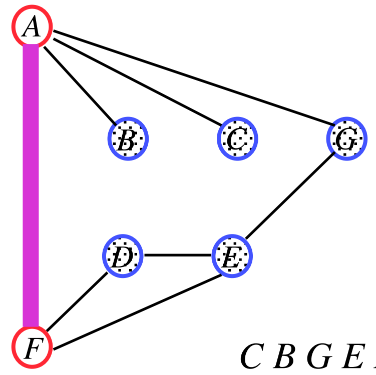
```
void traverse(Vertex startVertex) {
    Queue<Vertex> queue = new LinkedList<>();
    queue.add(startVertex);
    startVertex.visited = true;
    while (!queue.isEmpty()) {
        Vertex v = queue.remove();
        for (Edge e : v.adj) {
            Vertex w = e.dest;
            if (!w.visited) {
                queue.add(w);
                w.visited = true;
            }
        }
    }
}
```

Breadth-first traversal of a component

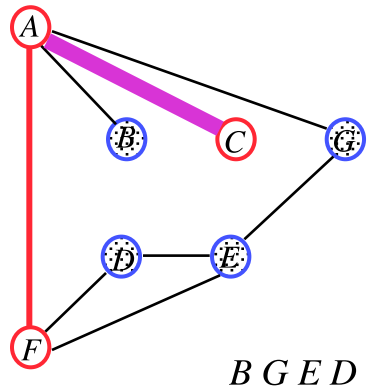
A: *F C B G*
B: *A*
C: *A*
D: *F E*
E: *G F D*
F: *A E D*
G: *E A*



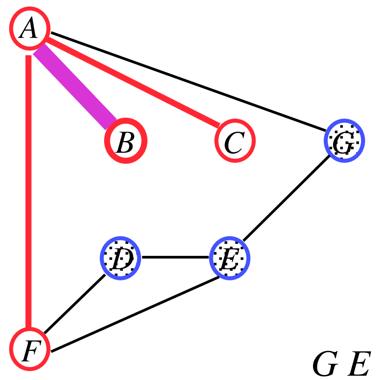
F C B G



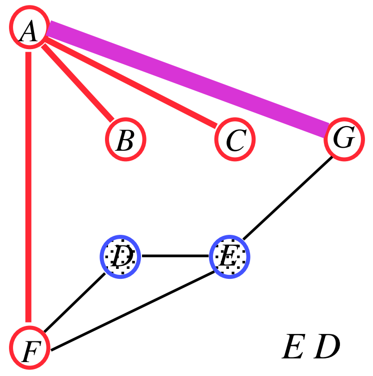
C B G E D



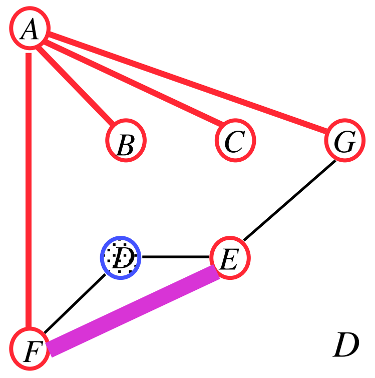
B G E D



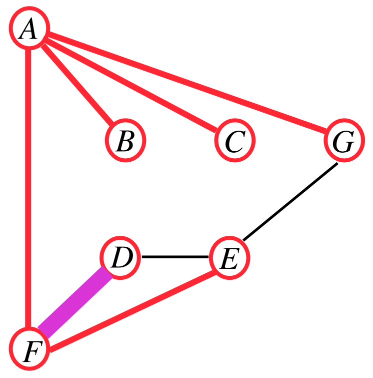
G E D



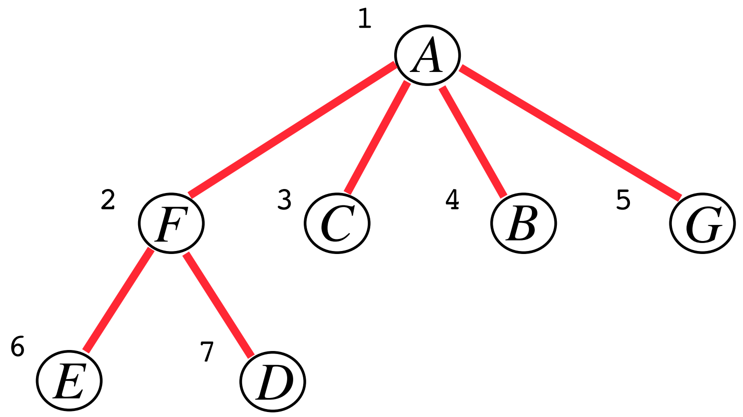
E D



D

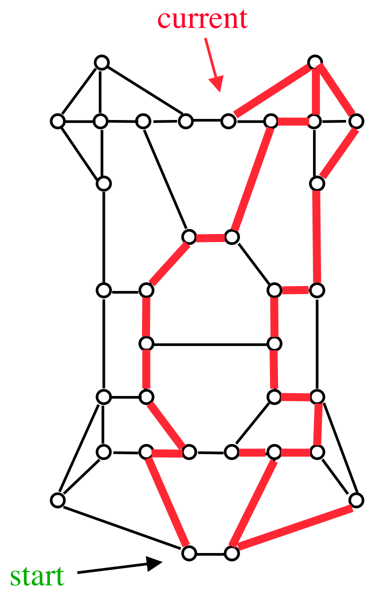


Breadth-first traversal of a component results in a breadth-first tree

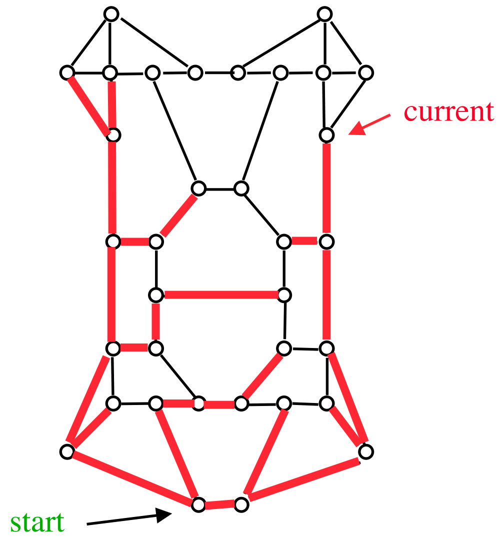


A breadth-first traversal of a connected graph represented by adjacency lists requires $O(|E|)$ time

Depth-first traversal versus breadth-first traversal



Depth-first



Breadth-first

Best-first traversal

If the queue is replaced by a priority queue, the graph will be traversed in *best-first order*.

```
Queue<Vertex> queue = new PriorityQueue<>();
```

Class `Vertex` should implement the `Comparable` interface, or the priority queue should rely on a supplied `Comparator` object.

$O(|E|)$ insertions and $O(|V|)$ removals; each takes $O(\log|V|)$ time for a heap-based priority queue.

Time complexity: $O((|V|+|E|)\log|V|)$

Shortest paths

The shortest path problem

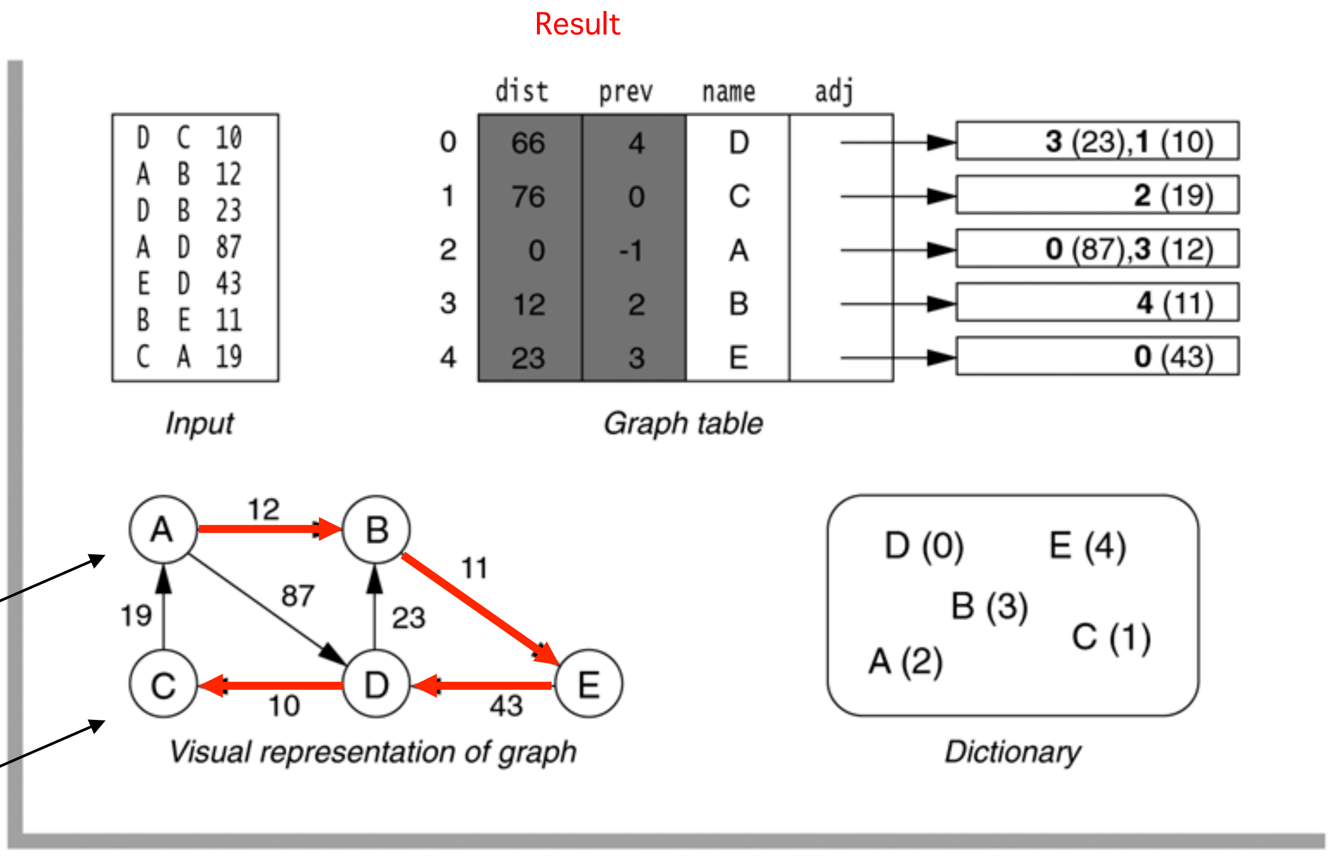
Find the shortest path from vertex *A* to vertex *B*

Unweighted shortest path (minimize the number of edges):
Use **breadth-first** traversal.
Traverse the graph starting at *A*, using a *queue*.

Weighted shortest path (find the “cheapest” path):
Use **best-first** traversal (**Dijkstra’s algorithm**):
Traverse the graph starting at *A*, using a *priority queue*.
The priority of each unvisited vertex is the cost of the currently cheapest path from *A* to that vertex.
Works only for graphs with non-negative weights.

figure 14.4

An abstract scenario of the data structures used in a shortest-path calculation, with an input graph taken from a file. The shortest weighted path from A to C is A to B to E to D to C (cost is 76).



Input

```
D  C  10
A  B  12
D  B  23
A  D  87
E  D  43
B  E  11
C  A  19
```

Graph table

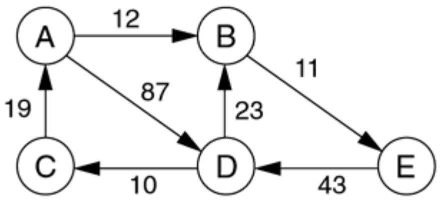
	dist	prev	name	adj
0	66	4	D	**3** (23),**1** (10)
1	76	0	C	**2** (19)
2	0	-1	A	**0** (87),**3** (12)
3	12	2	B	**4** (11)
4	23	3	E	**0** (43)

Dictionary

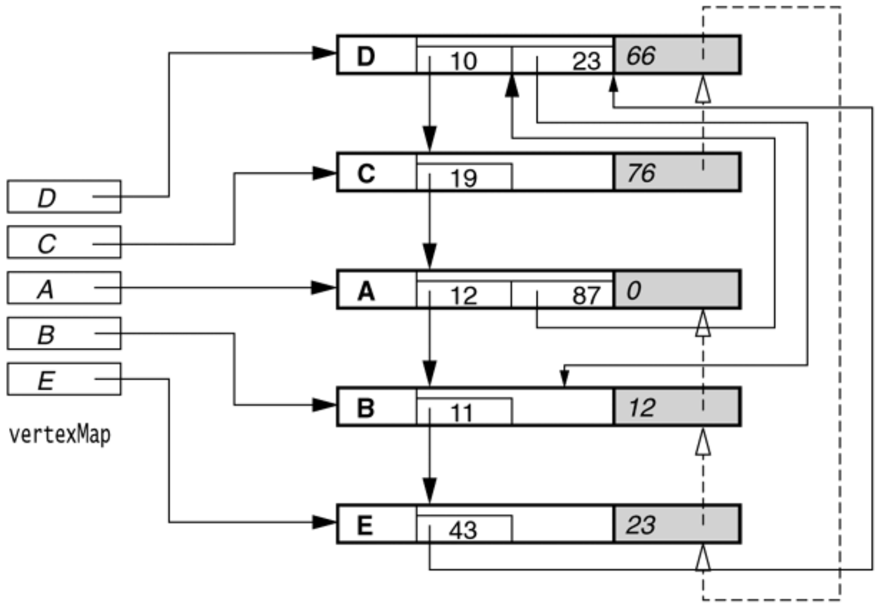
D (0) E (4) B (3) A (2) C (1)

```
D  C  10
A  B  12
D  B  23
A  D  87
E  D  43
B  E  11
C  A  19
```

Input



Visual representation of graph



Legend: Dark-bordered boxes are `Vertex` objects. The unshaded portion in each box contains the name and adjacency list and does not change when shortest-path computation is performed. Each adjacency list entry contains an `Edge` that stores a reference to another `Vertex` object and the edge cost. Shaded portion is `dist` and `prev`, filled in after shortest path computation runs.

Dark arrows emanate from `vertexMap`. Light arrows are adjacency list entries. Dashed arrows are the `prev` data member that results from a shortest-path computation.

figure 14.5

Data structures used in a shortest-path calculation, with an input graph taken from a file; the shortest weighted path from `A` to `C` is `A` to `B` to `E` to `D` to `C` (cost is 76).

Class `Edge`

```java
// Represents an edge in the graph.
class Edge
{
    public Vertex dest;       // Second vertex in Edge
    public double cost;       // Edge cost

    public Edge( Vertex d, double c )
    {
        dest = d;
        cost = c;
    }
}
```

figure 14.6

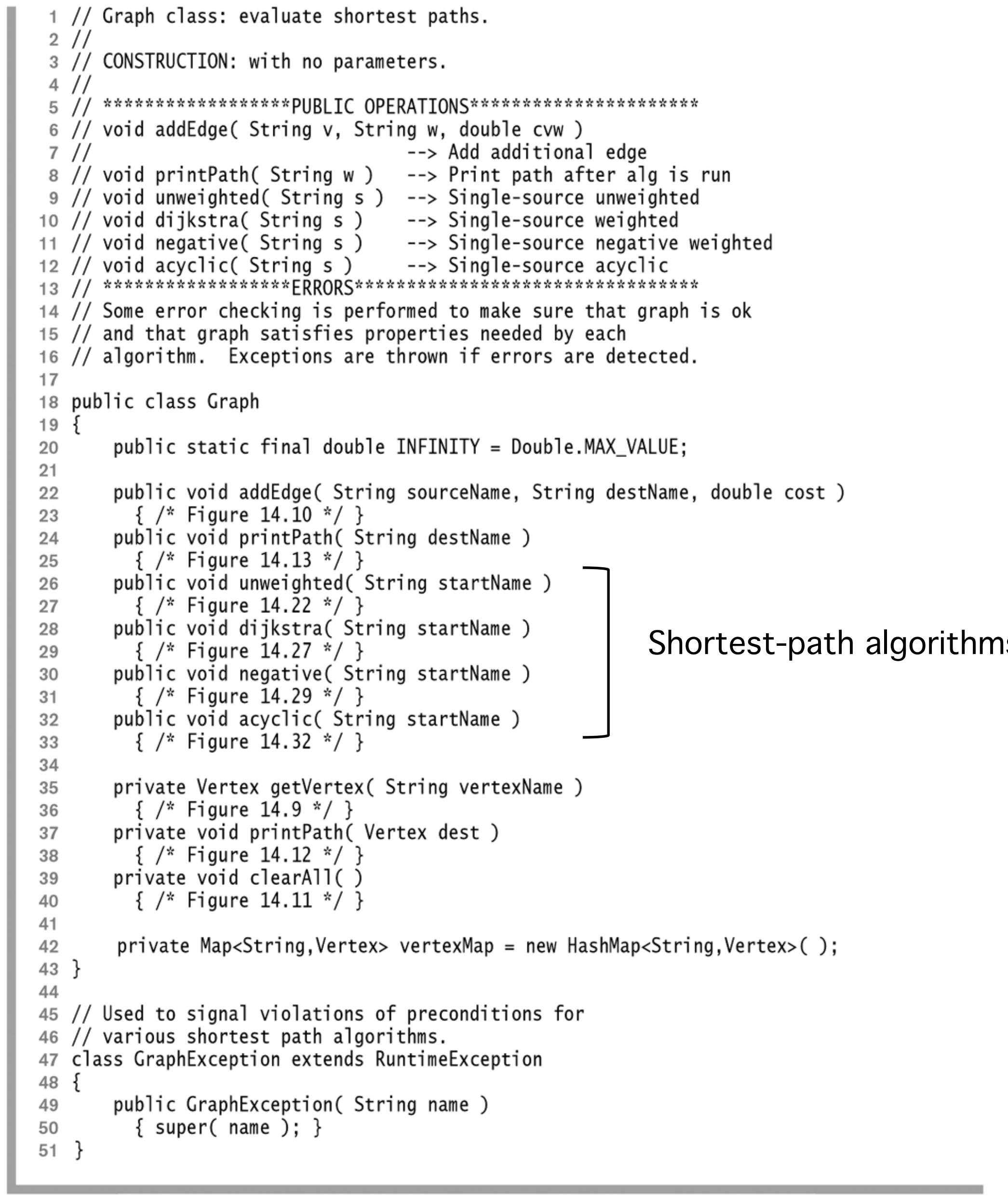
The basic item stored in an adjacency list

Class `Vertex`

```java
// Represents a vertex in the graph.
class Vertex
{
    public String     name;   // Vertex name
    public List<Edge> adj;    // Adjacent vertices
    public double     dist;   // Cost
    public Vertex     prev;   // Previous vertex on shortest path
    public int        scratch;// Extra variable used in algorithm

    public Vertex( String nm )
      { name = nm; adj = new LinkedList<Edge>( ); reset( ); }

    public void reset( )
     { dist = Graph.INFINITY; prev = null;            scratch = 0; }
}
```

figure 14.7

The `Vertex` class stores information for each vertex



```java
// Graph class: evaluate shortest paths.
//
// CONSTRUCTION: with no parameters.
//
// ******************PUBLIC OPERATIONS**********************
// void addEdge( String v, String w, double cvw )
//                               --> Add additional edge
// void printPath( String w )   --> Print path after alg is run
// void unweighted( String s )  --> Single-source unweighted
// void dijkstra( String s )    --> Single-source weighted
// void negative( String s )    --> Single-source negative weighted
// void acyclic( String s )     --> Single-source acyclic
// ******************ERRORS*********************************
// Some error checking is performed to make sure that graph is ok
// and that graph satisfies properties needed by each
// algorithm.  Exceptions are thrown if errors are detected.

public class Graph
{
    public static final double INFINITY = Double.MAX_VALUE;

    public void addEdge( String sourceName, String destName, double cost )
      { /* Figure 14.10 */ }
    public void printPath( String destName )
      { /* Figure 14.13 */ }
    public void unweighted( String startName )
      { /* Figure 14.22 */ }
    public void dijkstra( String startName )
      { /* Figure 14.27 */ }
    public void negative( String startName )
      { /* Figure 14.29 */ }
    public void acyclic( String startName )
      { /* Figure 14.32 */ }

    private Vertex getVertex( String vertexName )
      { /* Figure 14.9 */ }
    private void printPath( Vertex dest )
      { /* Figure 14.12 */ }
    private void clearAll( )
      { /* Figure 14.11 */ }

     private Map<String,Vertex> vertexMap = new HashMap<String,Vertex>( );
}

// Used to signal violations of preconditions for
// various shortest path algorithms.
class GraphException extends RuntimeException
{
    public GraphException( String name )
      { super( name ); }
}
```

Shortest-path algorithms

figure 14.8

The Graph class skeleton

```java
/**
 * If vertexName is not present, add it to vertexMap.
 * In either case, return the Vertex.
 */
private Vertex getVertex( String vertexName )
{
    Vertex v = vertexMap.get( vertexName );
    if( v == null )
    {
        v = new Vertex( vertexName );
        vertexMap.put( vertexName, v );
    }
    return v;
}
```

figure 14.9

The `getVertex` routine returns the `Vertex` object that represents `vertexName`, creating the object if it needs to do so

```
/**
 * Add a new edge to the graph.
 */
public void addEdge( String sourceName, String destName, double cost )
{
    Vertex v = getVertex( sourceName );
    Vertex w = getVertex( destName );
    v.adj.add( new Edge( w, cost ) );
}
```

figure 14.10

Add an edge to the graph

figure 14.11

Private routine for initializing the output members for use by the shortest-path algorithms

```
/**
 * Initializes the vertex output info prior to running
 * any shortest path algorithm.
 */
private void clearAll( )
{
    for( Vertex v : vertexMap.values( ) )
        v.reset( );
}
```

figure 14.12

A recursive routine for printing the shortest path

```
/**
 * Recursive routine to print shortest path to dest
 * after running shortest path algorithm. The path
 * is known to exist.
 */
private void printPath( Vertex dest )
{
    if( dest.prev != null )
    {
        printPath( dest.prev );
        System.out.print( " to " );
    }
    System.out.print( dest.name );
}
```

figure 14.13

A routine for printing the shortest path by consulting the graph table (see Figure 14.5)

```
/**
 * Driver routine to handle unreachables and print total cost.
 * It calls recursive routine to print shortest path to
 * destNode after a shortest path algorithm has run.
 */
public void printPath( String destName )
{
    Vertex w = vertexMap.get( destName );
    if( w == null )
        throw new NoSuchElementException( );
    else if( w.dist == INFINITY )
        System.out.println( destName + " is unreachable" );
    else
    {
        System.out.print( "(Cost is: " + w.dist + ") " );
        printPath( w );
        System.out.println( );
    }
}
```

```java
/**
 * A main routine that
 * 1. Reads a file (supplied as a command-line parameter)
 *    containing edges.
 * 2. Forms the graph.
 * 3. Repeatedly prompts for two vertices and
 *    runs the shortest path algorithm.
 * The data file is a sequence of lines of the format
 *    source destination.
 */
public static void main( String [ ] args )
{
    Graph g = new Graph( );
    try
    {
        FileReader fin = new FileReader( args[0] );
        BufferedReader graphFile = new BufferedReader( fin );

        // Read the edges and insert
        String line;
        while( ( line = graphFile.readLine( ) ) != null )
        {
            StringTokenizer st = new StringTokenizer( line );

            try
            {
                if( st.countTokens( ) != 3 )
                {
                    System.err.println( "Skipping bad line " + line );
                    continue;
                }
                String source  = st.nextToken( );
                String dest    = st.nextToken( );
                int    cost    = Integer.parseInt( st.nextToken( ) );
                g.addEdge( source, dest, cost );
            }
            catch( NumberFormatException e )
              { System.err.println( "Skipping bad line " + line ); }
         }
     }
     catch( IOException e )
       { System.err.println( e ); }

     System.out.println( "File read..." );
     System.out.println( g.vertexMap.size( ) + " vertices" );

     BufferedReader in = new BufferedReader(
                         new InputStreamReader( System.in ) );
     while( processRequest( in, g ) )
         ;
}
```

figure 14.14

A simple `main`

Input format:
source_name dest_name cost

```
/**
 * Process a request; return false if end of file.
 */
public static boolean processRequest( BufferedReader in, Graph g )
{
    String startName = null;
    String destName = null;
    String alg = null;

    try
    {
        System.out.print( "Enter start node:" );
        if( ( startName = in.readLine( ) ) == null )
            return false;
        System.out.print( "Enter destination node:" );
        if( ( destName = in.readLine( ) ) == null )
            return false;
        System.out.print( " Enter algorithm (u, d, n, a ): " );
        if( ( alg = in.readLine( ) ) == null )
            return false;

        if( alg.equals( "u" ) )
            g.unweighted( startName );
        else if( alg.equals( "d" ) )
            g.dijkstra( startName );
        else if( alg.equals( "n" ) )
            g.negative( startName );
        else if( alg.equals( "a" ) )
            g.acyclic( startName );

        g.printPath( destName );
    }
    catch( IOException e )
      { System.err.println( e ); }
    catch( NoSuchElementException e )
      { System.err.println( e ); }
    catch( GraphException e )
      { System.err.println( e ); }
    return true;
}
```

figure 14.15

For testing purposes, `processRequest` calls one of the shortest-path algorithms

Unweighted shortest path

(breadth-first traversal)

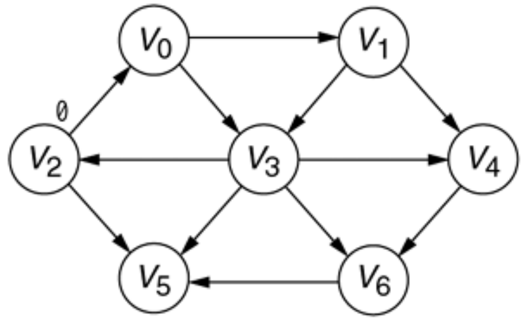


figure 14.16

The graph after the starting vertex has been marked as reachable in zero edges

figure 14.17

The graph after all the vertices whose path length from the starting vertex is 1 have been found

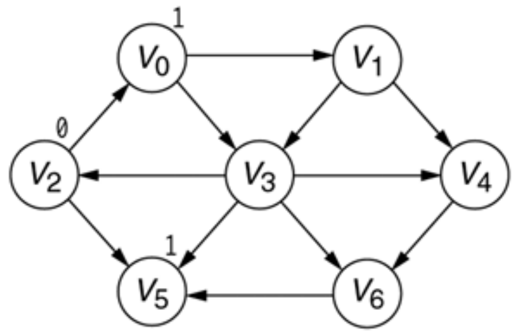


figure 14.18

The graph after all the vertices whose shortest path from the starting vertex is 2 have been found

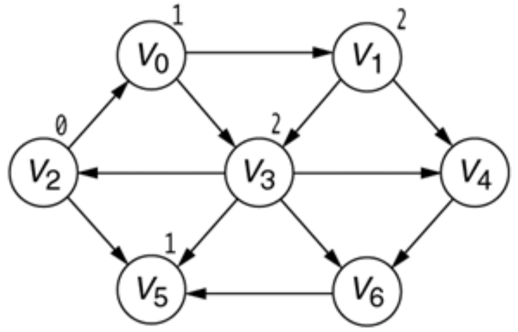
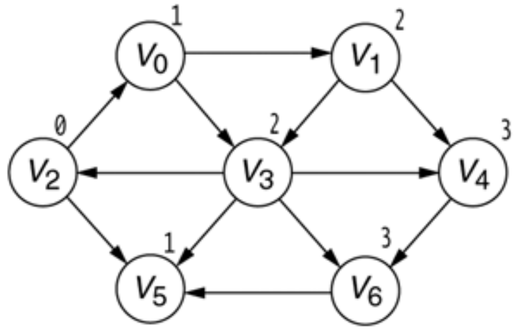


figure 14.19

The final shortest paths



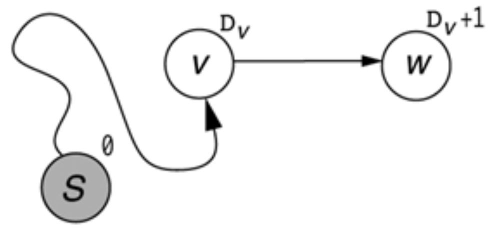


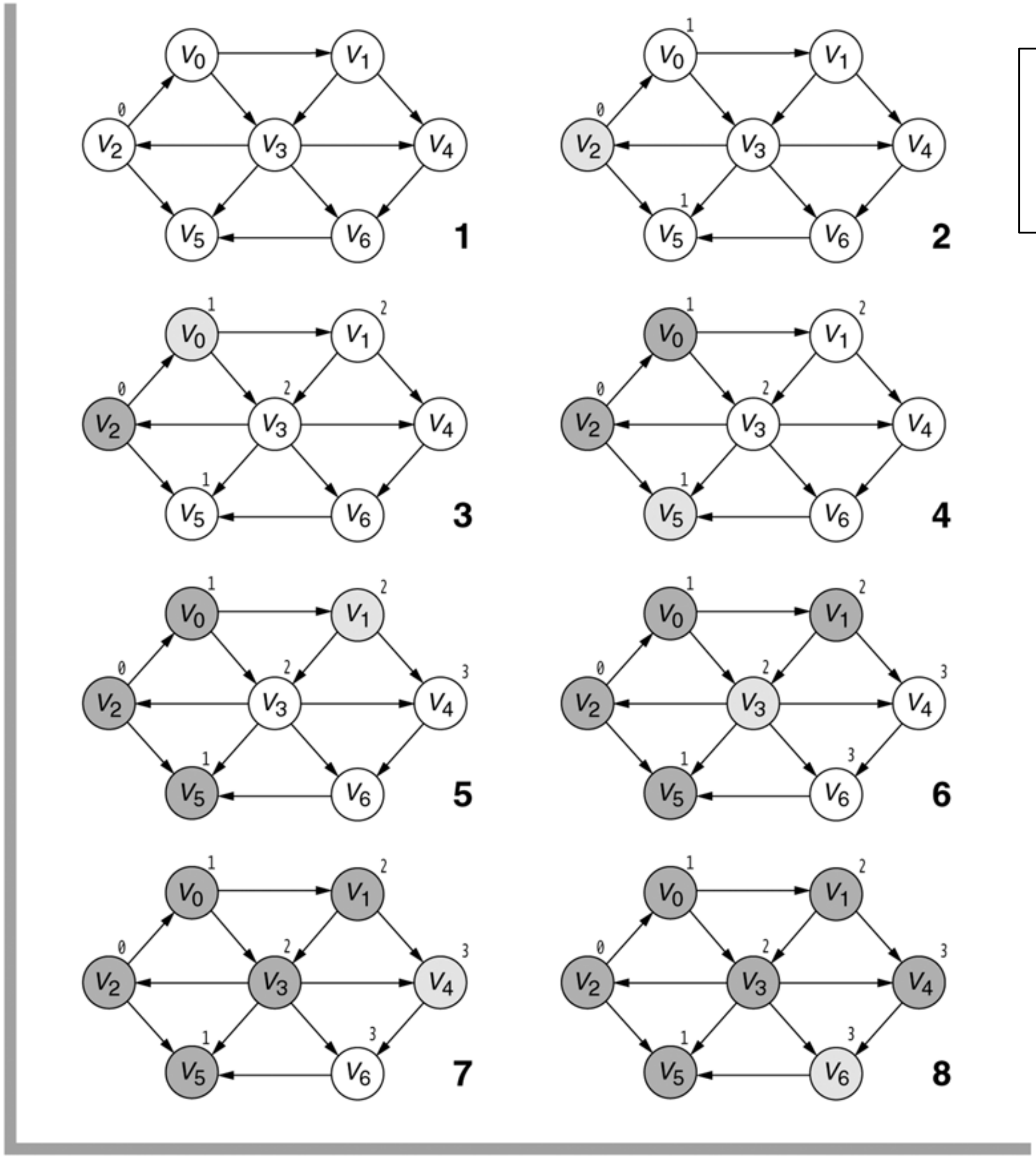
figure 14.20

If w is adjacent to v and there is a path to v, there also is a path to w.

(of cost $D_w = D_v+1$)

figure 14.21

Searching the graph in the unweighted shortest-path computation. The darkest-shaded vertices have already been completely processed, the lightest vertices have not yet been used as *v*, and the medium-shaded vertex is the current vertex, *v*. The stages proceed left to right, top to bottom, as numbered.



We maintain a *roving eyeball* that hops from vertex to vertex and is initially at V_2.

Roving eyeball
da. strejfende øjeæble

```java
/**
 * Single-source unweighted shortest-path algorithm.
 */
public void unweighted( String startName )
{
    clearAll( );

    Vertex start = vertexMap.get( startName );
    if( start == null )
        throw new NoSuchElementException( "Start vertex not found" );

    Queue<Vertex> q = new LinkedList<Vertex>( );
    q.add( start ); start.dist = 0;

    while( !q.isEmpty( ) )
    {
        Vertex v = q.remove( );

        for( Edge e : v.adj )
        {
            Vertex w = e.dest;

            if( w.dist == INFINITY )
            {
                w.dist = v.dist + 1;
                w.prev = v;
                q.add( w );
            }
        }
    }
}
```

figure 14.22

The unweighted shortest-path algorithm, using breadth-first search

Time complexity: $O(|E|)$

Positive weighted shortest path

(Dijkstra's algorithm, 1959)

E. W. Dijkstra, 1930-2002

For a given source vertex in the graph, the algorithm finds the path with lowest cost (i.e. the shortest path) between that vertex and every other vertex.

It can also be used for finding costs of shortest paths from a single vertex to a single destination vertex by stopping the algorithm once the shortest path to the destination vertex has been determined.

Dijkstra’s algorithm

Let the node at which we are starting be called the *initial node*. Let the *distance of node Y* be the distance from the initial node to *Y*.

1. Assign to every node a distance value. Set it to zero for our initial node and to infinity for all other nodes.
2. Mark all nodes as unvisited. Set initial node as current.
3. For the current node, consider all its unvisited neighbors and calculate their *tentative* distance (from the initial node). If this distance is less than the previously recorded distance (infinity in the beginning, zero for the initial node), overwrite the distance.
4. When we are done considering all neighbors of the current node, mark it as visited. A visited node will not be checked ever again; its distance recorded now is final and minimal.
5. If all nodes have been visited, finish. Otherwise, set the unvisited node with the smallest distance (from the initial node) as the next “current node” and continue from step 3.

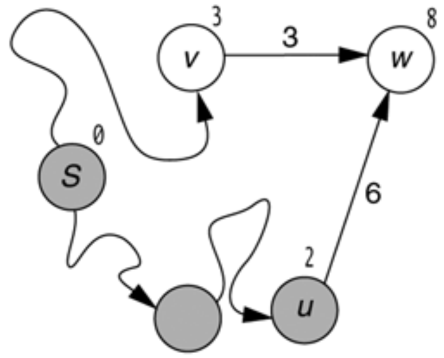
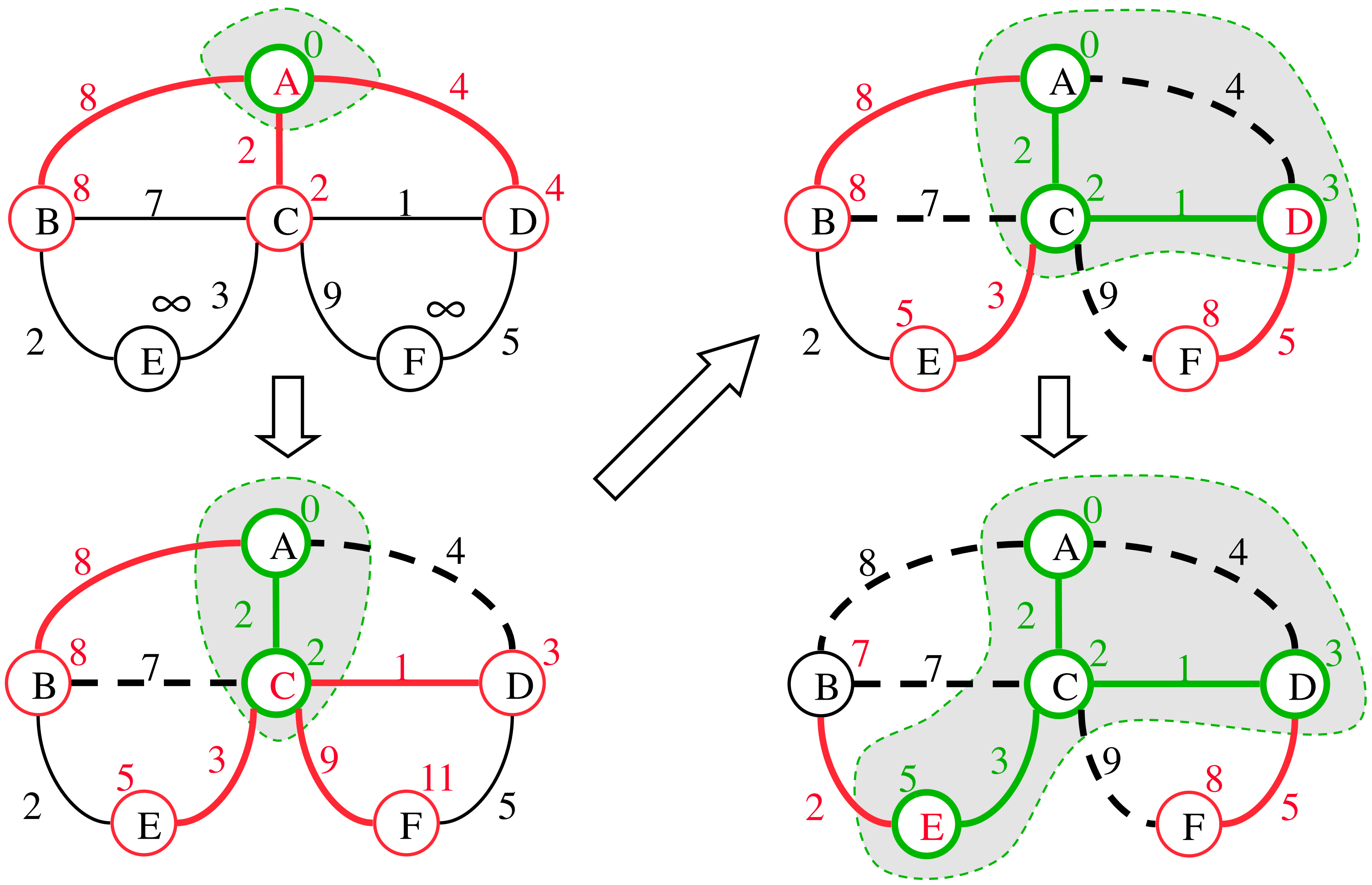


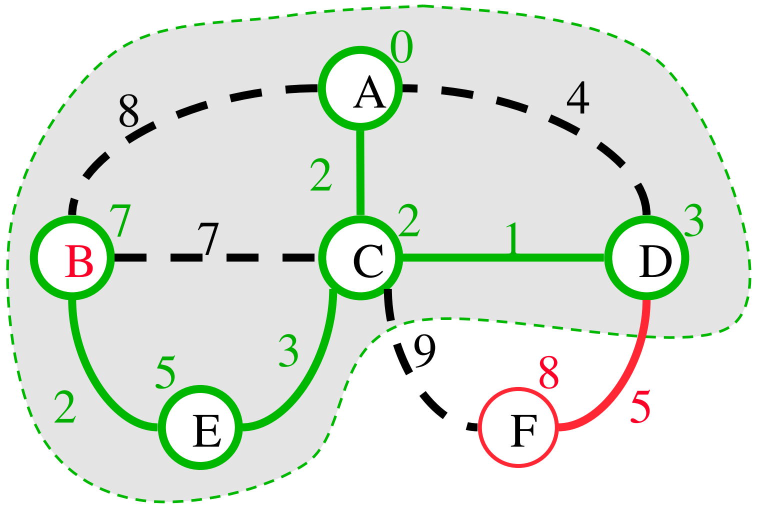
figure 14.23

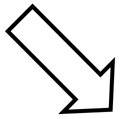
The eyeball is at v and w is adjacent, so D_w should be lowered to 6.

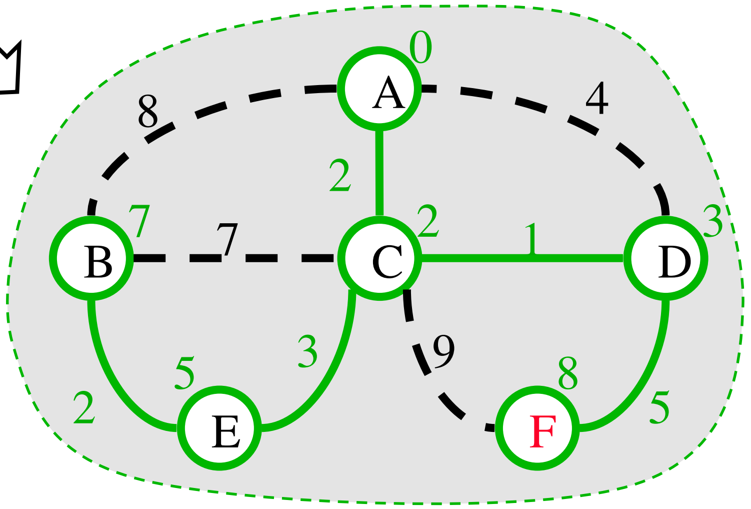
Example



Example continued







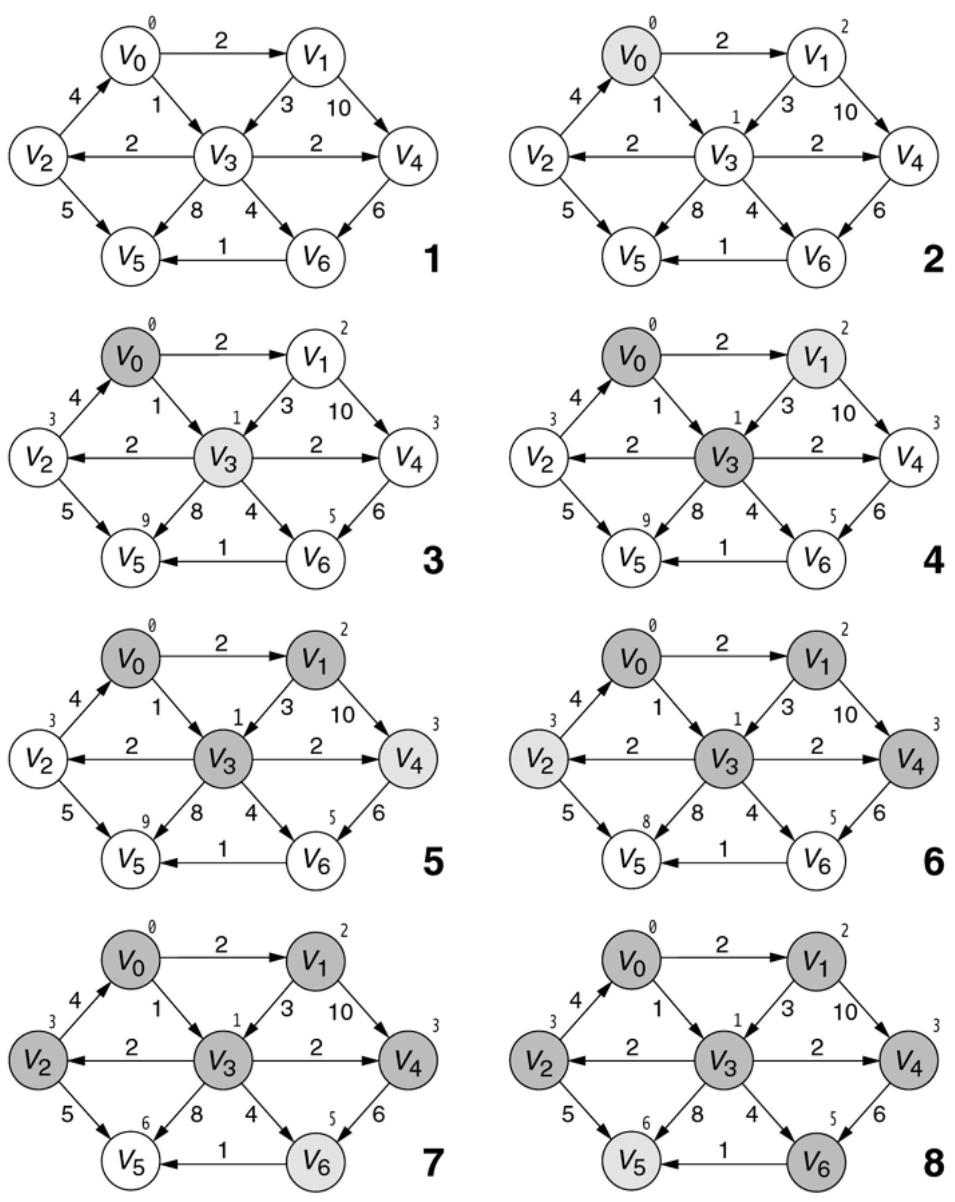
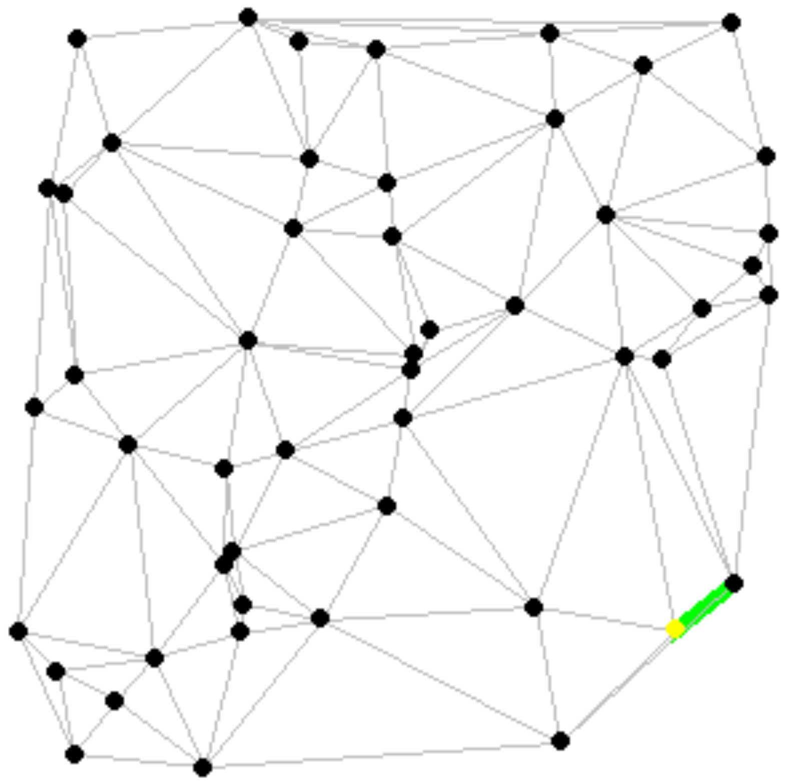


figure 14.25

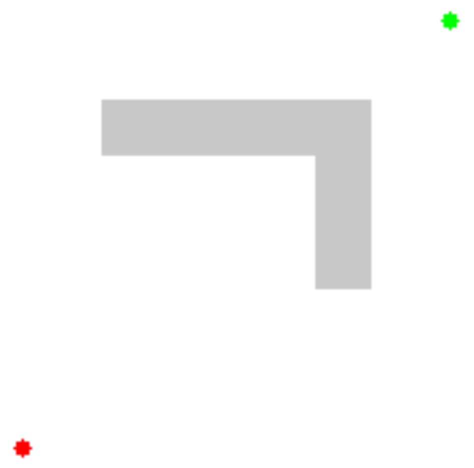
Stages of Dijkstra's algorithm. The conventions are the same as those in Figure 14.21.

Dijkstra's algorithm



www.combinatorica.com

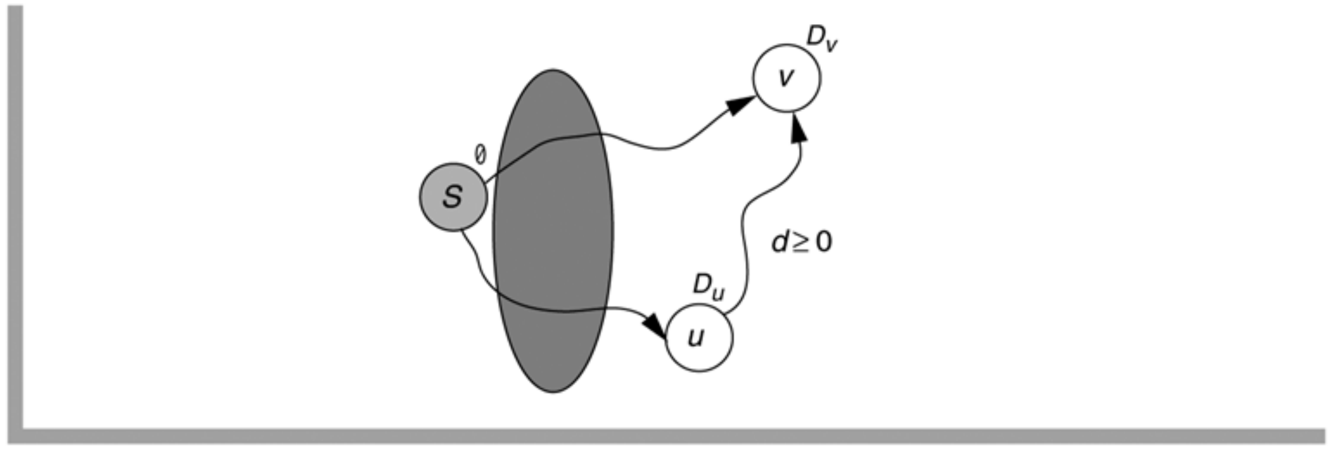
Dijkstra’s algorithm used for solving a robot planning problem



Proof of Dijkstra's algorithm

figure 14.24

If D_v is minimal among all unseen vertices and if all edge costs are nonnegative, D_v represents the shortest path.



Suppose there is a path from S to v of length less than D_v.
This path must go through a vertex u that has not yet been visited.
But since the length of the path from S to u, D_u, is less than D_v, we would have chosen u instead of v. Hence we have a contradiction.

Implementation of Dijkstra’s algorithm

```
void dijkstra(Vertex startVertex) {
    clearAll();
    PriorityQueue<Vertex> pq = new PriorityQueue<>();
    pq.add(startVertex); startVertex.dist = 0;
    while (!pq.isEmpty()) {
        Vertex v = pq.remove();
        for (Edge e : v.adj) {
            Vertex w = e.dest;
            if (v.dist + e.cost < w.dist) {
                w.dist = v.dist + e.cost;
                w.prev = v;
                pq.update(w); // error: no such method!
            }
        }
    }
}
```

`pq.update(w)`: If `w` is not in `pq`, then add `w` to `pq`; otherwise, update `pq` by reestablishing its ordering property. Unfortunately, the `update` method is not available in Java’s `PriorityQueue`.

Class `Path`

```java
// Represents an entry in the priority queue for Dijkstra's algorithm.
class Path implements Comparable<Path>
{
    public Vertex     dest;   // w
    public double     cost;   // d(w)

    public Path( Vertex d, double c )
    {
        dest = d;
        cost = c;
    }

    public int compareTo( Path rhs )
    {
        double otherCost = rhs.cost;

        return cost < otherCost ? -1 : cost > otherCost ? 1 : 0;
    }
}
```

figure 14.26

Basic item stored in the priority queue

```java
/**
 * Single-source weighted shortest-path algorithm.
 */
public void dijkstra( String startName )
{
    PriorityQueue<Path> pq = new PriorityQueue<Path>( );

    Vertex start = vertexMap.get( startName );
    if( start == null )
        throw new NoSuchElementException( "Start vertex not found" );

    clearAll( );
    pq.add( new Path( start, 0 ) ); start.dist = 0;

    int nodesSeen = 0;
    while( !pq.isEmpty( ) && nodesSeen < vertexMap.size( ) )
    {
        Path vrec = pq.remove( );
        Vertex v = vrec.dest;
        if( v.scratch != 0 )  // already processed v
            continue;

        v.scratch = 1;
        nodesSeen++;

        for( Edge e : v.adj )
        {
            Vertex w = e.dest;
            double cvw = e.cost;

            if( cvw < 0 )
                throw new GraphException( "Graph has negative edges" );

            if( w.dist > v.dist + cvw )
            {
                w.dist = v.dist + cvw;
                w.prev = v;
                pq.add( new Path( w, w.dist ) );
            }
        }
    }
}
```

figure 14.27

A positive-weighted, shortest-path algorithm: Dijkstra's algorithm

Time complexity:
$O(|E|\cdot\log|V|)$

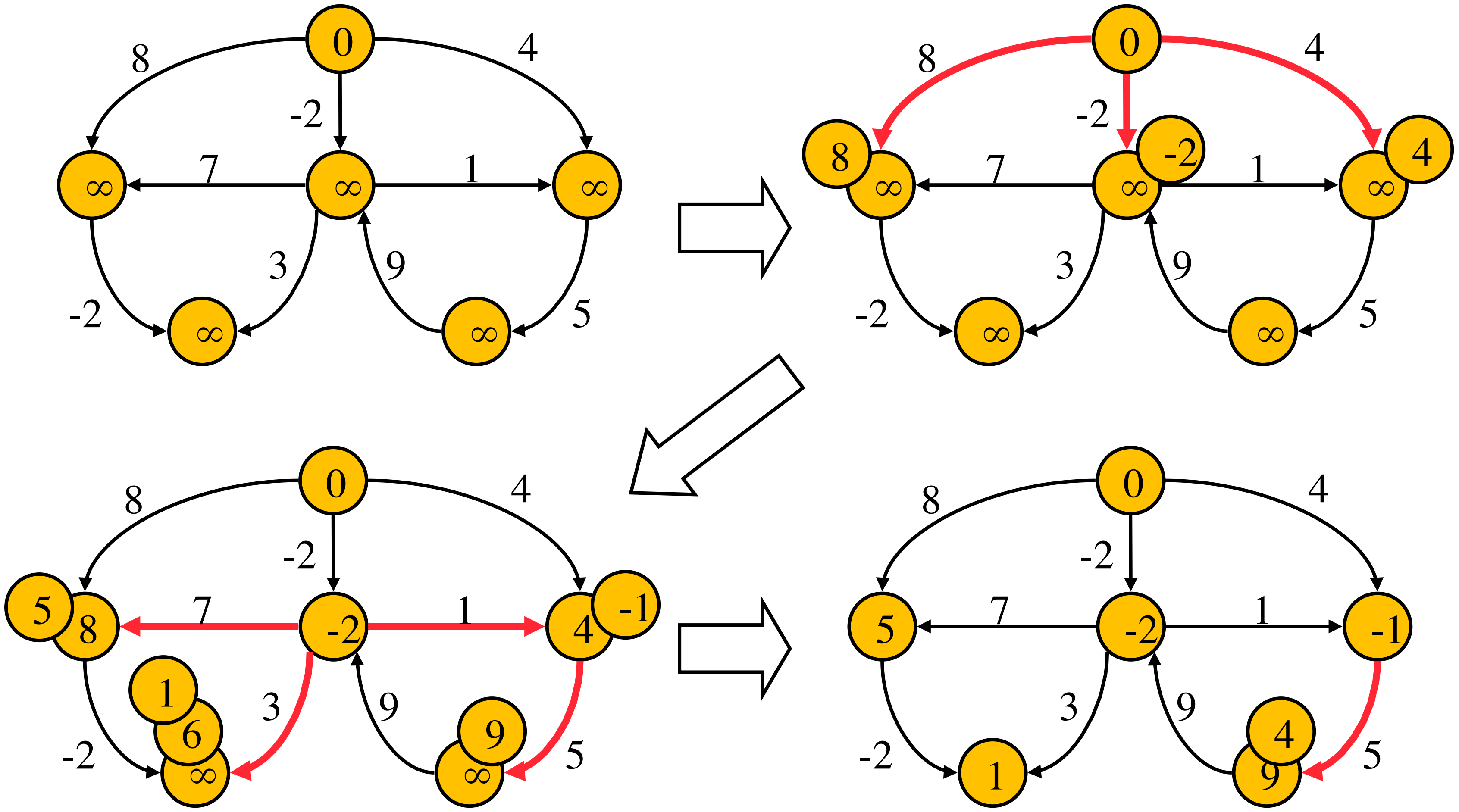
Negative-weighted shortest path

(The Bellman-Ford algorithm, 1958)

```
void bellmanFord(Vertex startVertex) {
     clearAll();
     startVertex.dist = 0;
     Collection<Vertex> vertices = vertexMap.values();
     for (int i = 1; i < vertices.size(); i++) {
         for (Vertex v : vertices) {
             for (Edge e : v.adj) {
                 Vertex w = e.dest;
                 if (v.dist + e.cost < w.dist) {
                     w.dist = v.dist + e.cost;
                     w.prev = v;
                 }
             }
         }
     }
}
```

Iteration *i* finds all shortest paths from `startVertex` that uses *i* or fewer edges.
Time complexity: $O(|E|\cdot|V|)$

Bellman-Ford example



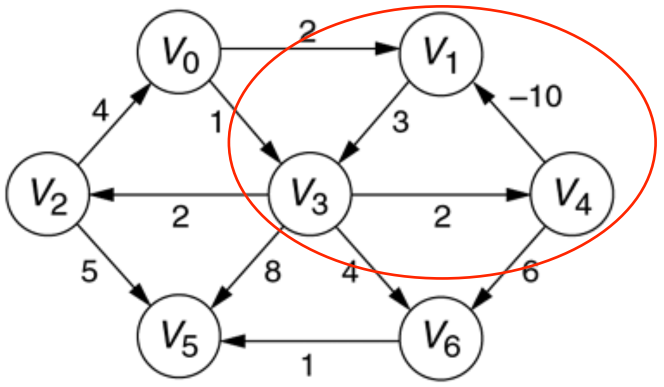


figure 14.28

A graph with a negative-cost cycle

Check for negative-cost cycles (add this code after the loop):

```
for (Vertex v : vertices) {
    for (Edge e : v.adj) {
        Vertex w = e.dest;
        if (v.dist + e.cost < w.dist)
            error("Negative-cost cycle detected");
    }
}
```

```java
/**
 * Single-source negative-weighted shortest-path algorithm.
 */
public void negative( String startName )
{
    clearAll( );

    Vertex start = vertexMap.get( startName );
    if( start == null )
        throw new NoSuchElementException( "Start vertex not found" );

    Queue<Vertex> q = new LinkedList<Vertex>( );
    q.add( start ); start.dist = 0; start.scratch++;

    while( !q.isEmpty( ) )
    {
        Vertex v = q.removeFirst( );
        if( v.scratch++ > 2 * vertexMap.size( ) )
            throw new GraphException( "Negative cycle detected" );

        for( Edge e : v.adj )
        {
            Vertex w = e.dest;
            double cvw = e.cost;

            if( w.dist > v.dist + cvw )
            {
                w.dist = v.dist + cvw;
                w.prev = v;
                  // Enqueue only if not already on the queue
                if( w.scratch++ % 2 == 0 )
                    q.add( w );
                else
                    w.scratch--;  // undo the enqueue increment
            }
        }
    }
}
```

figure 14.29

A negative-weighted, shortest-path algorithm: Negative edges are allowed.

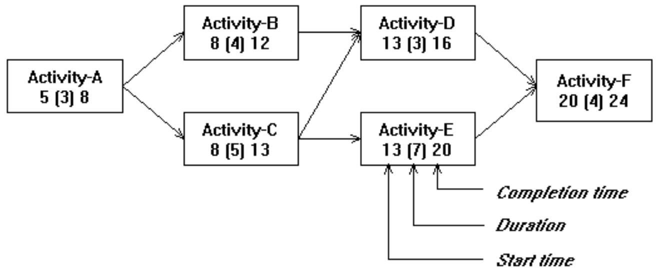
`v.scratch` is odd when vertex `v` is on the queue. `v.scratch/2` tells us how many times `v` has left the queue.

An edge can dequeue at most $O(|V|)$ times.
Time *complexity*: $O(|E|\cdot|V|)$

DAGs

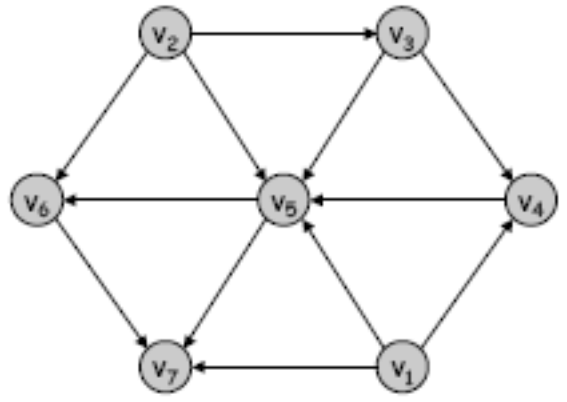
An oriented graph without cycles is called a **DAG** (**D**irected **A**cyclic **G**raph).

A DAG may, for instance, be used for modeling an activity network. Directed edges are used to specify that some activities must be finished before an activity can start.

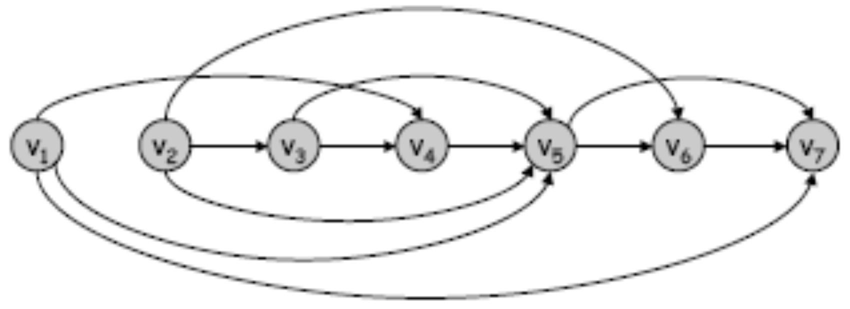


Topological sorting

The vertices of a DAG can be ordered so that if there is a path from *u* to *v*, then *v* appears after *u* in the ordering. This is called a **topological sort** of the graph.



a DAG



a topological ordering

Topological ordering: All directed edges point from left to right
[not necessarily unique]

A topological sorting algorithm

(1) Create an empty queue

(2) Choose a vertex without any ingoing edges

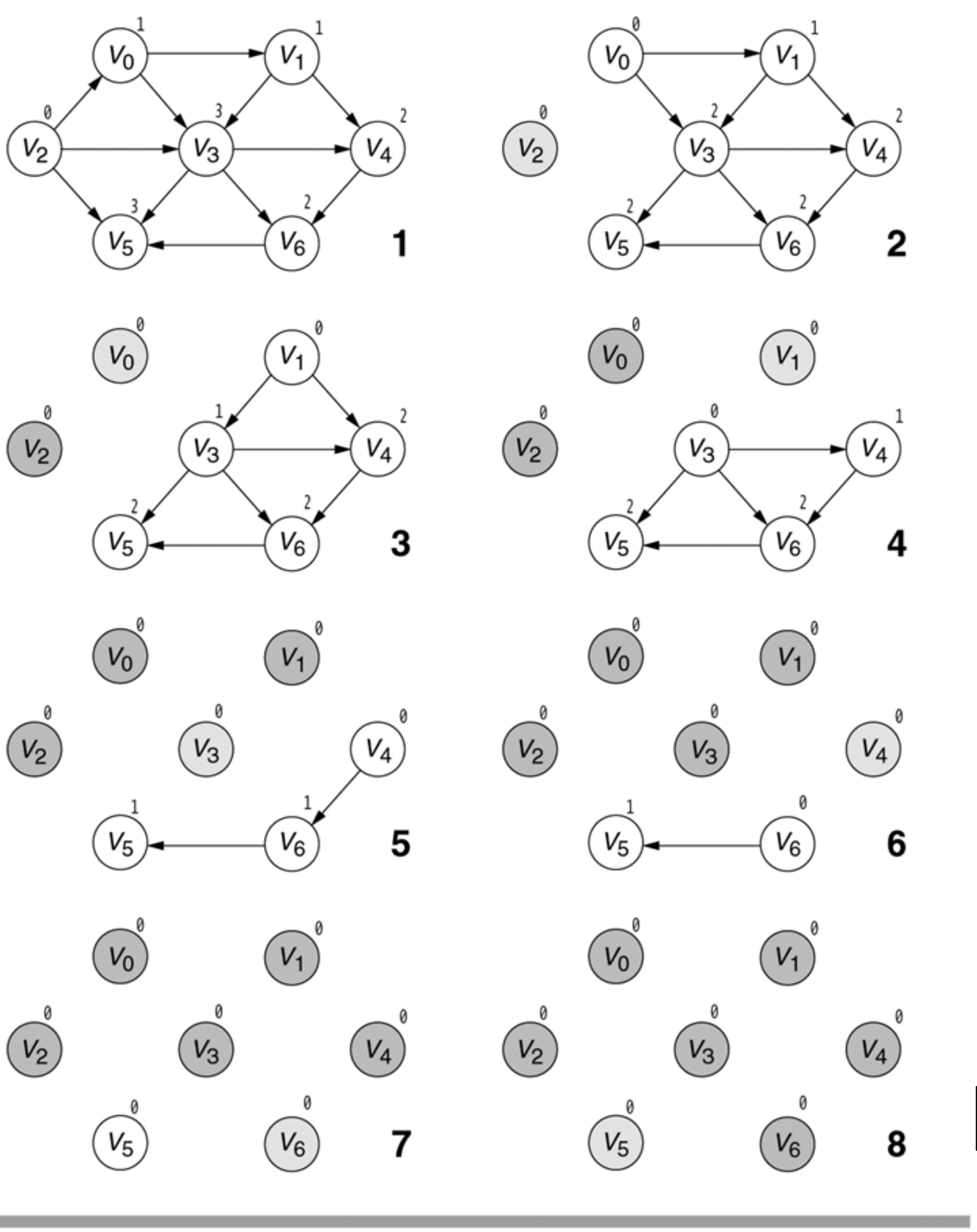
(3) Insert the vertex in the queue. Remove the vertex and all its outgoing edges from the graph.

(4) Repeat (2) and (3) while the graph is not empty

Now the queue contains the vertices in topological order

figure 14.30

A topological sort. The conventions are the same as those in Figure 14.21.



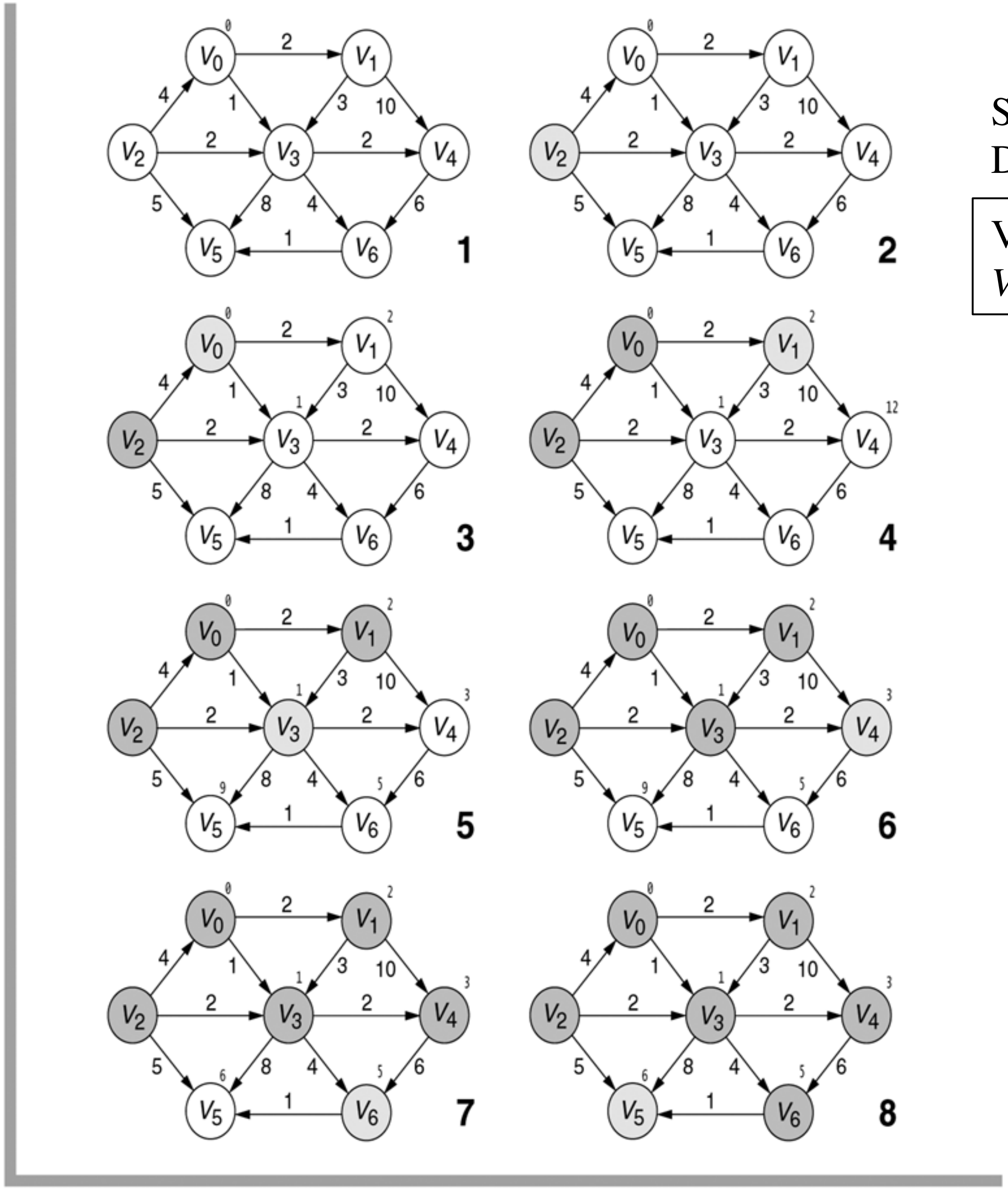
$V_2\ V_0\ V_1\ V_3\ V_4\ V_6\ V_5$

Java implementation

```
List<Vertex> tologicalOrder() {
    Collection<Vertex> vertices = vertexMap.values();
    for (Vertex v : vertices)
        v.scratch = 0;        // v's indegree = 0
    for (Vertex v : vertices)
        for (Edge e : v.adj)
            e.dest.scratch++;
    Queue<Vertex> q = new LinkedList<>();
    for (Vertex v : vertices)
        if (v.scratch == 0)
            q.add(v);
    List<Vertex> result = new ArrayList<>();
    int iterations = 0;
    while (!q.isEmpty() && ++iterations <= vertices.size()) {
        Vertex v = q.remove();
        result.add(v);
        for (Edge e : v.adj)
            if (--e.dest.scratch == 0)
                q.add(e.dest);
    }
    return iterations == vertices.size() ? result : null;
}
```

figure 14.31

The stages of acyclic graph algorithm. The conventions are the same as those in Figure 14.21.



Shortest path for a DAG

Visit order:
V_2 V_0 V_1 V_3 V_4 V_6 V_5

```java
/**
 * Single-source negative-weighted acyclic-graph shortest-path algorithm.
 */
public void acyclic( String startName )
{
    Vertex start = vertexMap.get( startName );
    if( start == null )
        throw new NoSuchElementException( "Start vertex not found" );

    clearAll( );
    Queue<Vertex> q = new LinkedList<Vertex>( );
    start.dist = 0;

      // Compute the indegrees
    Collection<Vertex> vertexSet = vertexMap.values( );
    for( Vertex v : vertexSet )
        for( Edge e : v.adj )
            e.dest.scratch++;

      // Enqueue vertices of indegree zero
    for( Vertex v : vertexSet )
        if( v.scratch == 0 )
            q.add( v );

    int iterations;
    for( iterations = 0; !q.isEmpty( ); iterations++ )
    {
        Vertex v = q.remove( );

        for( Edge e : v.adj )
        {
            Vertex w = e.dest;
            double cvw = e.cost;

            if( --w.scratch == 0 )
                q.add( w );

            if( v.dist == INFINITY )
                continue;

            if( w.dist > v.dist + cvw )
            {
                w.dist = v.dist + cvw;
                w.prev = v;
            }
        }
    }

    if( iterations != vertexMap.size( ) )
        throw new GraphException( "Graph has a cycle!" );
}
```

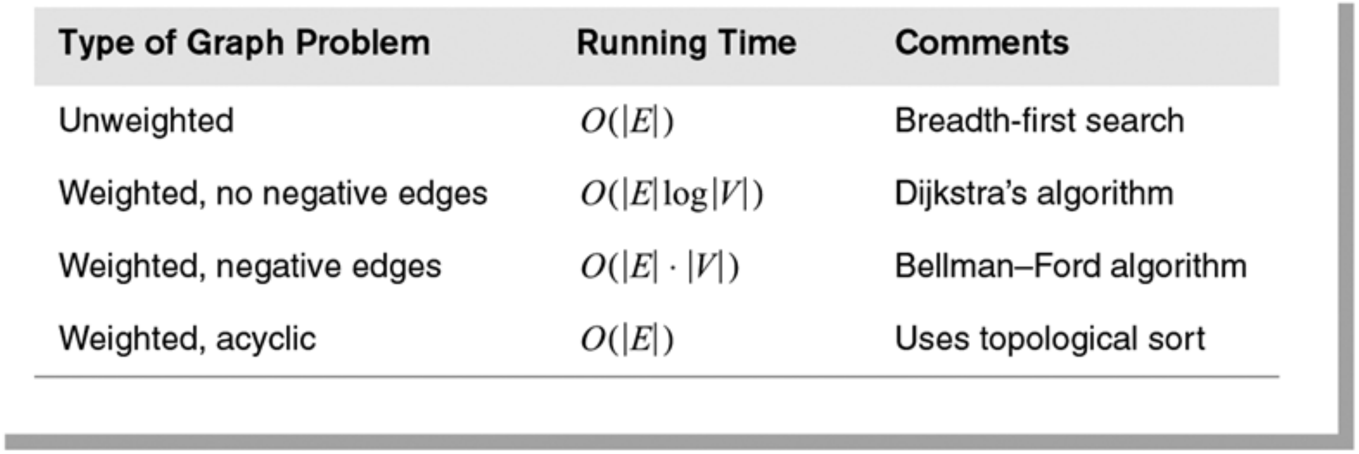
figure 14.32

A shortest-path algorithm for acyclic graphs

Uses topological sort

Time complexity: $O(|E|)$

Complexity of shortest path algorithms



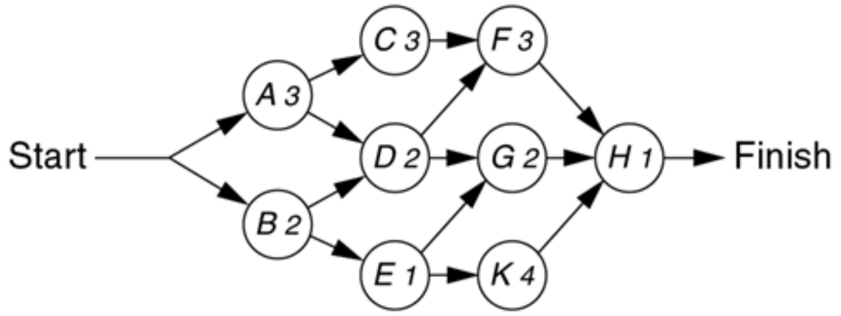
Type of Graph Problem	Running Time	Comments
Unweighted	$O(\|E\|)$	Breadth-first search
Weighted, no negative edges	$O(\|E\| \log \|V\|)$	Dijkstra's algorithm
Weighted, negative edges	$O(\|E\| \cdot \|V\|)$	Bellman–Ford algorithm
Weighted, acyclic	$O(\|E\|)$	Uses topological sort

figure 14.38

Worst-case running times of various graph algorithms

figure 14.33

An activity-node graph



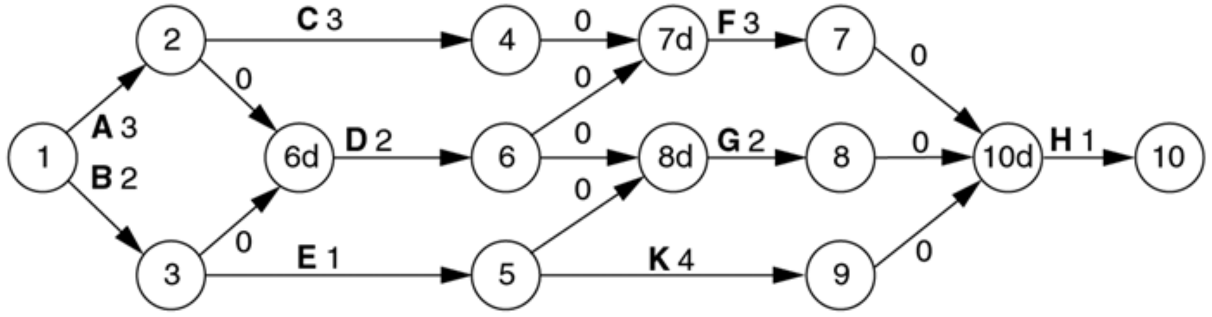


figure 14.34

An event-node graph

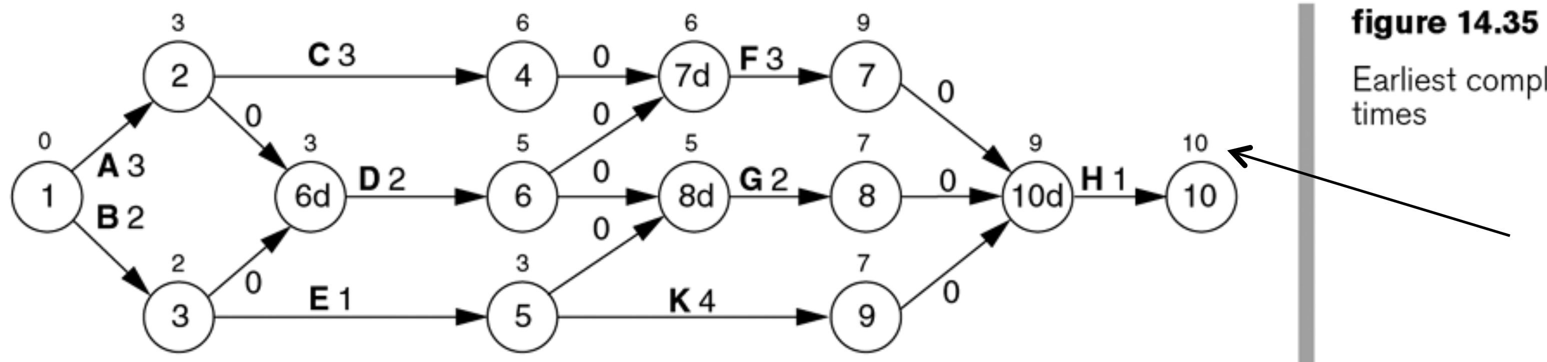
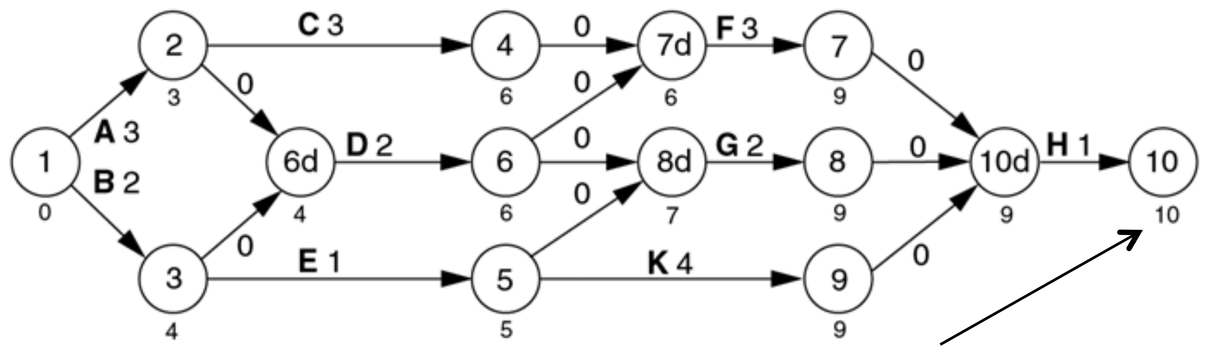


figure 14.35

Earliest completion times

figure 14.36

Latest completion times



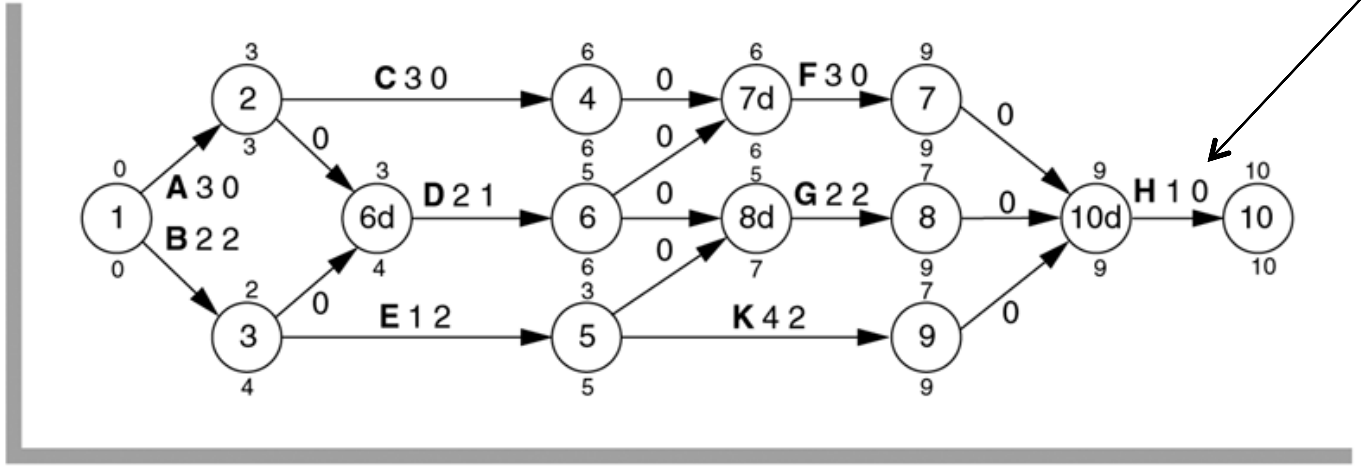
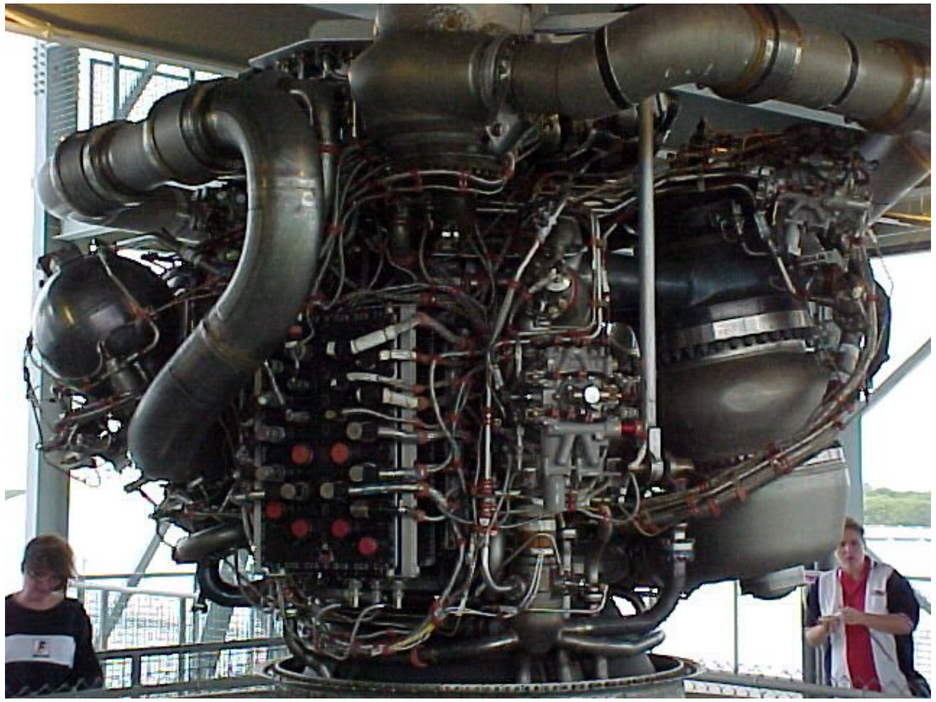


figure 14.37

Earliest completion time, latest completion time, and slack (additional edge item)

Some activities have zero slack. These are critical activities that must be finished on schedule. A path consisting entirely of zero-slack edges is a **critical path**.

Problem complexity



Problem complexity

For a large class of important problems no fast solution algorithms are known.

An **efficient algorithm**: running time is limited by some polynomial
[$O(n^c)$]

A problem that can be solved by a efficient algorithm is said to be **easy**.

An **inefficient** algorithm: running time grows at least exponentially
[$\Omega(c^n)$]

A problem is said to be **hard** or intractable if there does not exist a polynomial-time algorithm for solving the problem.

Examples of hard problems

- **The traveling salesman problem**
  A salesman must visit *N* cities. Find a travel route that minimizes his costs.

- **Job scheduling**
  A number of jobs of varying duration are to be executed on two identical machines before a given deadline. Is it possible to meet the deadline?

- **Satisfiability**
  Is it possible to determine if the variables in a Boolean expression can be assigned in such a way as to make the expression evaluate to true?

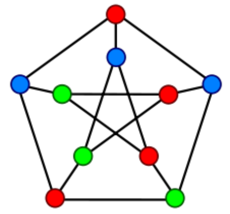
$$(a \vee b) \wedge (\neg a \vee b)$$

More examples of hard problems

- **Longest path**
  Find the longest simple path between two vertices of a graph.

- **Partitioning**
  Given at set of integers. Is it possible to partition the set into two subsets so that the sum of the elements in each of the two subsets is the same?

- **3-coloring**
  Is it possible to color the vertices of a graph by only three colors such that no two adjacent vertices have the same color?



NP-complete problems

For none of these problems do we know an algorithm that solves the problem in polynomial time.

All experts are convinced that such algorithms do not exist. However, this has not yet been proved.

The problems belong to the class of problems called **NP-complete** problems.

NP-completeness

An NP-complete problem is a problem that can be solved in *polynomial* time on a **nondeterministic** machine.

A nondeterministic machine has the wonderful ability to make the correct choice in any situation where a choice is to be made.

A usual deterministic machine may be used to simulate correct choices in exponential time by trying each possible choice.

If only one NP-complete problem can be solved in polynomial time, every NP-complete problem can be solved in polynomial time.

Decidability

Undecidable problems are decision problems which no algorithm can decide.

Examples:

- Prove that an algorithm always terminates (the stop problem)
- Decide if a formula in the predicate logic is valid
- Decide if two syntax descriptions define the same language

Termination?

(a)

```
while (x != 1)
    x = x - 2;
```

(b)

```
while (x != 1)
    if (x % 2 == 0)
        x = x / 2;
    else
        x = 3 * x + 1;
```

Collatz sequences:

12, 6, 3, 10, 5, 16, 8, 4, 2, 1

9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1

Collatz conjecture (1937): No matter what number you start with, you will always eventually reach 1.
The conjecture has still not been proven!

Termination?

(continued)

(c)

```
for (int x = 3; ; x++)
for (int a = 1; a <= x; a++)
for (int b = 1; b <= x; b++)
for (int c = 1; c <= x; c++)
for (int n = 3; n <= x; n++)
    if (Math.pow(a,n) + Math.pow(b,n) == Math.pow(c,n))
        System.exit(0);
```

The program terminates, if and only if Fermat's last theorem is false.

For $n \geq 3$, no three positive integers a, b, and c can satisfy $a^n + b^n = c^n$.

P. de Fermat (1601-65)

The theorem was proven in 1995

The halting problem

It is impossible to design an algorithm that for any algorithm can decide if it terminates.

Proof (by contradiction):

Assume there exists a method `terminates(p)`, which for any method `p` returns `true` if `p` terminates; otherwise, `false`.

Now define:

```
void p() {
   while (terminates(p)) /* do nothing */;
}
```

What is the result of the call `terminates(p)`?

CAUTION

YOU HAVE REACHED THE

LAST PAGE

OF THE INTERNET

TURN OFF YOUR BROWSER AND GO BACK TO WORK
THERE'S NOTHING ELSE TO SEE HERE