THE FOUR SIDES AND THE AREA

Oblique Light on the Prehistory of Algebra

By

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The present essay traces the career of a particular mathematical problem—to find the side of a square from the sum of its four sides and the area—from its first appearance in an Old Babylonian text until it surfaces for the last time in the same unmistakeable form during the Renaissance in Luca Pacioli's and Pedro Nunez' works. The problem turns out to belong to a non-scholarly tradition carried by practical geometers, together with other simple quasi-algebraic "recreational" problems dealing with the sides, diagonals and areas of squares and rectangles. This "mensuration algebra" (as I shall call it) was absorbed into and interacted with a sequence of literate mathematical cultures: the Old Babylonian scribal tradition, early Greek so-called metric geometry, and Islamic *al-jabr*. The article explores how these interactions inform us about the early history of algebraic thinking.

As far as possible I have referred for detailed documentation to earlier publications, in particular to my analysis of Babylonian "algebra" and its reflections in later traditions. In cases where documentation is not discussed in depth elsewhere I have still tried to be concise, but none the less felt obliged to present at least an outline of the full argument.

I. An Old Babylonian "square problem"

A famous cuneiform mathematical text (BM 13901)[1] contains as its Nº 23 the following problem

> In a surface, the f[o]u[r fronts and the surf]ace I have accumulated, 41´40´´.
> 4, the f[ou]r fronts, yo[u inscr]ibe. The i g i of 4 is 15´.
> 15´ to 41´40´´ [you r]aise: 10´25´´ you inscribe.
> 1, the projection, you append: 1°10´25´´ makes 1°5´ equilateral.
> 1, the projection, which you have appended, you tear out: 5´ to two
> you repeat: 10´ n i n d a n confronts itself.

The text was written in the Old Babylonian period, that is, between 2000 BC and 1600 BC, and probably during the eighteenth century BC. Originally, it appears to have contained 24 problems of apparently algebraic character dealing with one or more squares and their sides. In its present state, the tablet is damaged, though most problems can be safely reconstructed.

The translation is meant to render the terminology as precisely as possible, and follows principles which I have developed for the translation of Babylonian "algebra".[2] In the present context, only a few words' explanation can be made. Numbers, first of all, are rendered in the degree-minute-second notation, which means that 1°10´25´´ is to be read $1+\frac{10}{60}+\frac{25}{60\cdot 60}$. (One should remember that the original text contains no indicators of absolute order of magnitude, merely the sequence 1 10 25.) “Accumulating” (Akkadian *kamārum*[3]) is a genuine addition of numbers, where both addends loose their identity and merge into a sum; as here, it may be used for additions with no concrete interpretation (length plus area). “Appending” (*waṣābum*), on the other hand, is a concrete additive operation, where one entity (one may think as example of one’s own bank account) is augmented by another (the interests of the year—actually labelled “the appended” in Akkadian) without changing its identity (it remains *my* account). Appending possesses an inverse operation “tearing out” (*nasāḫum*); the other (“comparative”) subtractive operation “*a* exceeds *b* by *x*” (*a eli b x iter*) is only used for concretely meaningful comparisons, and is thus no real inverse of “accumulating”.

The “i g i” of a number *n* is its reciprocal as listed in a table of reciprocals. When having to divide by *n*, the Babylonians would multiply by i g i *n*, using an operation labelled “raising” (*našûm*)—probably best to be explained as “calculation [of something] by means of multiplication”; other multiplicative operations are “*a* steps of *b*” (*b* a - r à *a*), designating the multiplication of number by number in a multiplication table; “repeating to *n*” (*ina n eṣēpum*), which is indeed an *n*-fold concrete repetition; and “making *a* and *b* hold each other” (the most plausible reading of *a ù b šutakūlum*), which means arranging the lines [with lenghts] *a* and *b* as sides of a rectangle [whose area will then be $a{\cdot}b$]. A variant of the latter operation is “making *a* confront itself” (*a šutamḫurum*), which means making *a* the side of a square. The reverse of the latter operation is to find out what “makes [the area] *B* equilateral” (*B* í b - s i$_8$), that is, what length *a* will be the side if *B* is formed as a square (arithmetically: $a = \sqrt{B}$). The “projection” (*wasītum*) 1, finally, is a line segment of length 1 which, projecting orthogonally from another line segment [with the length] *a*, transforms it into a rectangle [with the area] $1{\cdot}a = a$. Lengths are measured in the unit n i n d a n (1 n i n d a n = 6 m) and areas in s a r (= n i n d a n^2)

With this is mind, we can understand the text. The first line tells that we are dealing with a surface (details in the grammar seem indeed to suggest *a field*). The sum of the measuring numbers for *the four* sides (not just *four times* the side) and the area is 41´40´´. In modern notation, if *s* is the length of the side, this corresponds to the equation $s^2 + 4s = 41´40´´$, which is the reason that this and similar Babylonian problems are generally regarded as algebra. The second line prepares a division by 4, which takes place in line 3; in our equation, this division would express itself in a transformation into $(^s/_2)^2 + 1{\cdot}s = 41´40´´/4 = 10´25´´$. The addition of 1 in line 4 would tell *us* that $(^s/_2)^2 + 2{\cdot}1{\cdot}(^s/_2) + 1 = 1°10´25´´$; finding the equilateral corresponds to the transformation $^s/_2 + 1 = \sqrt{1°10´25´´} = 1°5´$, leading us to the further conclusion that $^s/_2 = 5´$—and finally $s = 10´$.

The numerical steps of the solution are thus meaningful when seen in the perspective of symbolic algebra, yet the use of the term “projection” (and the addition of a mere “1” instead of

“1^2” in line 4, which is an otherwise compulsory Babylonian practice) tells us that the Babylonian calculator operated in a very different representation—see Figure 1: Each of the four sides was thought of as provided with a projection (that is, a “projecting width”) 1[4] and thus represented by a rectangle $s \times 1$; the surface was a square $s \times s$; and the sum was hence represented by a cross-shaped configuration. When the Babylonian scribe divided by 4 in lines 2–3, what he did was to single out one fourth of this configuration, for example, the gnomon in the upper left corner.

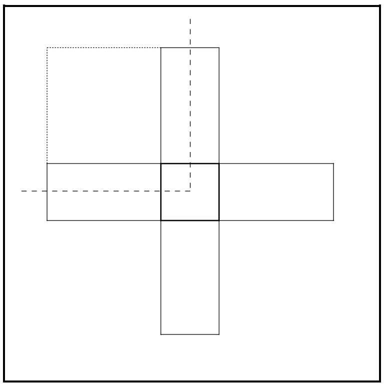

Figure 1. The procedure of BM 13901, N° 23.

The addition of “1 the projection” calls for a general commentary: *We* think of a square as *being* (for instance) 4 square feet and *having* the side 2 feet (knowing that, strictly speaking, the square is a complex configuration which can equally well be characterized by any of these parameters). The Babylonians, on their part, thought of the square as *being* 2 feet and *having* an area 4 square feet.[5] Appending “1 the projection” thus means fitting in the square contained by the gnomon, each of whose sides is indeed the projection. Thereby the gnomon is completed as a square with known area 1+10´25´´ = 1°10´25´´, which is “made equilateral” by √1°10´25´´ = 1°5´. From this, the projection (this time, according to *our* distinction, viewed as the side of the completing square) is torn out, leaving 5´ as the width of the gnomon leg. “Repeating” this to two, that is, uniting it with its mirror image, produces the side of the original square, that which “confronts itself”.

This “cut-and-paste procedure” is “naive” in the sense that everything can be “seen” immediately to be correct (whenever the word is used in the following it is to be read in this technical sense and never as “gullible”). There is no attempt to prove, for example, that the gnomon *is* a rectangular gnomon and contains precisely a square; such “critical” reflection (in a quasi-Kantian sense) had to wait until Euclid. But the procedure *can* be seen to be correct (and can be transformed into a “critical” proof without difficulty), and is thus justification and algorithm in one (as is the stepwise transformation of a modern algebraic equation). It is also “analytical” in the sense that the unknown side is treated as if it were known until it can be isolated from the complex relation in which it is entangled. If algebra is understood primarily as the application of analysis (as François Viète would have it), the method is clearly algebraic in nature. But if algebra is a science of *number* (or, post-Noether, generalized number) by means of abstract symbols, the Old Babylonian “algebra of measurable line segments” is *not algebra*. This proviso should be kept mind in the following when I drop the quotes for reasons of stylistic simplicity, speaking simply of *Babylonian algebra*.

Many features of the present problem are shared by the Old Babylonian “algebra” texts in general: The distinction between two additive operations—that is, operations which when translated into modern equations become additions; the analogous distinction between two different subtractive and no less than four different multiplicative operations; and the use of naive cut-and-paste geometry in procedures which are their own immediate justification. Other features, however, single out the problem of “the four sides and the area” as a remarkable exception.

If by Q we designate the quadratic area and by s the corresponding side (Q_i and s_i, $i = 1, 2, ...$ when several squares are involved); by ${}_4s$ “the four” sides of a square); if □(a) stands for the area of the square on the line segment a and ⊏⊐(a,b) for that of the rectangle “held” by a and b, the tablet contains the following problems (n` stands for $n{\cdot}60^1$):

1. $Q+s$ = 45´
2. $Q-s$ = 14`30
3. $Q-{}^1/_3Q+{}^1/_3s$ = 20´
4. $Q-{}^1/_3Q+s$ = 4`46°40´
5. $Q+s+{}^1/_3s$ = 55´
6. $Q+{}^2/_3s$ = 35´
7. $11Q+7s$ = 6°15´
8. Q_1+Q_2 = 21´40´´, s_1+s_2 = 50´ (reconstructed)
9. Q_1+Q_2 = 21´40´´, $s_2 = s_1$+10´
10. Q_1+Q_2 = 21°15´, $s_2 = s_1-{}^1/_7s_1$
11. Q_1+Q_2 = 28°15´, $s_2 = s_1+{}^1/_7s_1$
12. Q_1+Q_2 = 21´40´´, ⊏⊐(s_1,s_2) = 10´
13. Q_1+Q_2 = 28´20´´, $s_2 = {}^1/_4s_1$
14. Q_1+Q_2 = 25´25´´, $s_2 = {}^2/_3s_1$+5´
15. $Q_1+Q_2+Q_3+Q_4$ = 27´5´´, $(s_2,s_3,s_4) = ({}^2/_3,{}^1/_2,{}^1/_3)s_1$
16. $Q-{}^1/_3s$ = 5´
17. $Q_1+Q_2+Q_3$ = 10`12°45´, $s_2 = {}^1/_7s_1$, $s_3 = {}^1/_7s_2$
18. $Q_1+Q_2+Q_3$ = 23´20´´, $s_2 = s_1$+10´, $s_3 = s_2$+10´
19. Q_1+Q_2+□(s_1-s_2) = 23´20´´, s_1+s_2 = 50´
20. [missing]
21. [missing]
22. [missing]
23. ${}_4s+Q$ = 41´40´´
24. $Q_1+Q_2+Q_3$ = 29´10´´, $s_2 = {}^2/_3s_1$+5´, $s_3 = {}^1/_2s_2$+2´30´´

We observe that Nº 23 is the only problem referring to “the four” sides of a square. It is also the only problem mentioning the sides before the area. It is certainly not the only normalized mixed second-degree problem dealing with a single square, but all the others refer to a general method (in semi-modern terms: halving the number of sides, squaring this half, etc.). In geometric terms, a sides are expressed as ⊏⊐(a,s); this rectangle is bisected, and the total area Q + 2⊏⊐(½a,s) is transformed into a gnomon which is then completed; etc.—see Figure 2. The procedure of Nº 23, on the other hand, depends critically on the number 4; already at this point we may observe that this use of an amazing and elegant but non-generalizable solution makes the problem look more like a riddle than like a normal piece of mathematics (Babylonian or modern); so does, in fact, the presence of precisely *those four* sides which really belong to the square, instead of an arbitrary (and thus virtually general) multiple.

Other differences are no less striking. All remaining problems tell that they deal with squares by using the term which at one time designates the quadratic configuration and the length of the side; Nº 23 is alone in stating at first that it deals with "a surface" or (probably) "a field". It is also alone in using the term translated here as "front" (*pūtum*), an Akkadian term corresponding to Sumerian s a g , the "width" of a rectangle. In normal algebraic problems the Sumerian term is compulsory; the use of a word belonging to the spoken vocabulary of surveyors indicates that we are supposed to think of a real piece of land.

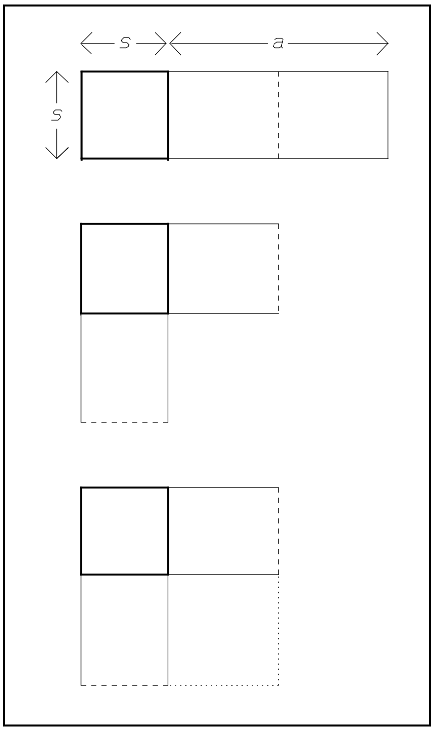

Figure 2. The "normal" procedure of BM 13901 for the solution of $Q+\alpha s = C$.

Even the solution is uncommon. Other problems of the tablet dealing with a single square have the side equal to 30´ (or 30), except for one case of 20´. These are indeed the standard values of square sides in Old Babylonian algebra problems, which may have to do with the roundness of these numbers in the sexagesimal place value notation used in mathematics teaching (30´ = $^1/_2$, 20´ = $^1/_3$).[6] All other cases where 10´ is found are caused by the use of other favourites (ratios 4 and 7, differences 10´ and 5´). Only Nº 23 (at least among those problems which are conserved) is constructed from the side 10´ as a *deliberate* choice. And only Nº 23 tells the unit of the result, as if it were to be entered into a cadastral or similar document (cf. note 6).

The final puzzling feature does not concern the problem itself but its place: Apart from Nº 16 (which can be suspected of having been displaced), problems of the type $\alpha Q \pm \beta s = C$ occur in the beginning of the tablet, and the neighbours of Nº 23 are considerably more complex. It seems as if the difference in method as reflected in the contrast between Figure 1 and Figure 2 was understood as a difference between mathematical genres.

II. The Proofs of **al-jabr**

No other Babylonian mathematical tablet contains a problem involving “the four” sides of a square or making use of the peculiar method of Figure 1. In order to find parallels we have to make a jump to the early ninth century CE.

This was the moment when the Khalif al-Maʾmūn asked al-Khwārizmī to put together a treatise covering those parts of the field *al-jabr wa’l-muqābalah* that were either “brilliant” (*laṭīf*) or practically useful.[7] Al-Khwārizmī is thus not to be considered the inventor of *al-jabr* (Latinized as *algebra*), and as we can read in a treatise by the slightly later Thābit ibn Qurrah[8], it was practiced by a group of “*al-jabr* people,” evidently some kind of professional calculators. Yet within another generation or two, Abū Kāmil would regard it as al-Khwārizmī’s discipline—and al-Khwārizmī appears indeed (together with his contemporary ibn Turk, from whose work only a fragment is extant) to have reshaped the discipline, in particular the treatment of second-degree problems, which was its core.[9]

Figure 3. Al-Khwārizmī’s second proof. From B. B. Hughes, “Gerard of Cremona’s Translation of al-Khwārizmī’s *Al-Jabr*,” p. 238.

The problem which we translate as $x^2 + 10x = 39$ would be formulated as follows by the *al-jabr* people: *A treasure together with 10 roots equals 39 dirhems*. Fundamentally, the problem thus tells that an unknown amount of money (the “treasure” or *māl*—more precisely “property”) together with 10 times its [square] root (*jaḏr*) equals 39 dirhems (strictly speaking, the correct translation is hence $y + 10\sqrt{y} = 39$). They would find the root by an unexplained rule: You halve [the number of] roots (which gives 5), multiply it by itself (25), add this to the dirhems (64), take the root (8), and subtract the half of the [number of] roots. Thus the root is 3, and the treasure is 9.

This rule is given by al-Khwārizmī and repeated by Thābit ibn Qurrah. It can safely be assumed to belong to the inherited lore of the group. Al-Khwārizmī’s most important innovation was to give a geometrical proof that the traditional rule (and the corresponding rules for the cases *Treasure and number equal roots* and *Roots and number equal treasure*) was correct. As in the Greek texts translated by al-Khwārizmī’s colleagues at the Baghdad court, points and areas are labelled by letters in these proofs. In essence, however, they only differ from the cut-and-paste proofs which we have encountered above by being more precisely argued and hence less naive.

For the case *The treasure together with 10 roots equals 39 dirhems*, two different proofs are given. The second corresponds directly to the rule, and is made on a diagram similar to Figure 2 (see Figure 3, which renders Gherardo of Cremona's translation). The first corresponds to a procedure that differs from the one whose correctness is to be proved: 10 is divided by 4 ($2^1/_2$), squared ($6^1/_4$), multiplied by 4 (25), and added to 39. The diagram (see Figure 4) corresponds to that of Figure 1. There is no reason *within* al-Khwārizmī's text to bring a diagram so obviously at odds with what is to be proved (elsewhere, he confesses no particular infatuation with symmetry). If the diagram is there it must be because it comes first to his mind, or because he expects it to come first to the reader's mind. It must hence be supposed to have been familiar either to al-Khwārizmī or to his "model reader"—not from the *al-jabr* but from some other tradition. (It is indeed also more naive in style than the following proofs.)

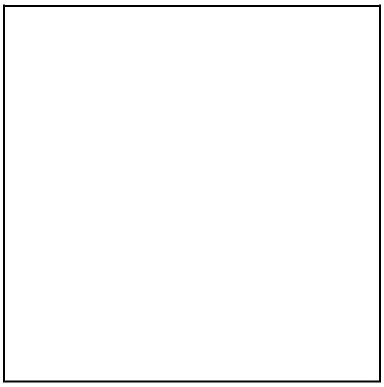

Figure 4. Al-Khwārizmī's first proof. From B. B. Hughes, "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*," p. 237.

III. Abū Bakr's "mensuration algebra"

This conjecture is confirmed by another treatise, a *Liber mensurationum* written by one unidentified Abū Bakr. According to terminological criteria the work will be grossly contemporary with al-Khwārizmī's.[10] No manuscript of the Arabic text is known, but a careful Latin translation was made by Gherardo of Cremona.[11] Moreover, as we shall see, Leonardo Fibonacci has used the work in his *Pratica geometrie*.

Formally, the work deals with practical geometry, and some of it really does. Thus, in the beginning of the first chapter it is told how, given the side of a square, the area and the diagonal can be calculated. Then, however, Abu Bakr goes on with "brilliant" problems of no or scarce practical interest and mostly asking for some kind of algebraic treatment; all in all, the initial chapter (on squares) contains the following problems:

1. $s = 10$: Q?
2. $s = 10$: d?
3. $s+Q = 110$: s?
4. $_4s+Q = 140$: s_u?
5. $Q-s = 90$: s?
6. $Q-{_4s} = 60$: s_u?
7. $_4s = {^2/_5}\cdot Q$: s_u?
8. $_4s = Q$: s_u?
9. $_4s-Q = 3$: s_u? (Both solutions are given)
10. $d = \sqrt{200}$; s?
11. $d = \sqrt{200}$; Q?

12. ${}_4s+Q = 60$: s_u?
13. $Q-3s = 18$: s?
14. ${}_4s = {}^3/_8 \cdot Q$: s_u?[12]
15. $Q/d = 7\frac{1}{2}$: s_u?
16. $d-s = 4$: s?
17. $d-s = 5$ (no question, refers to the previous case).
18. $d = s_u+4$: s? (no reference is made to Nº 16).
19. $Q/d = 7^1/_{14}$: s?, d?

Here, Q again denotes the area and s the side of the square; d is the diagonal, ${}_4s$ stands for "[the sum of] its four sides" (or merely "its sides," meaning the same), and s_u for "each of its sides" (below, A shall be used about the area of a rectangle, and l_1 and l_2 about its sides). The next chapters (rectangles regarded as "quadrates longer on one side," and rhombi) are similarly weighted toward algebraic problems; only then come chapters dominated by genuine geometrical calculation (and clearly related to the Alexandrian/Heronian tradition). In order to possess a name for this particular kind of quasi-algebra I shall speak about "mensuration algebra"—dropping again the quotes in the following for stylistic reasons, even though the objections to this characterization of the technique as algebra *tout court* are even stronger than in the case of the scribe school discipline (cf. note 22).

Returning to the chapter on the square we observe, firstly, that "the four sides and the area" turns up as Nº 4, and again with a different numerical parameter as Nº 12—the sides being once more mentioned first (in the *Liber mensurationum* this is the common usage). Secondly, that *all* problems involving sides except Nº 13 deal with *the* side or *the four* sides; later on, the sides of rectangles also invariable turn up in geometrically meaningful company—the shorter or the longer side alone, these two together, or all four together (similarly also the diagonals of rhombi). Thirdly, that the standard square has a side equal to 10, the only real exceptions being Nºs 8–9 and 12–13.[13]

Abū Bakr solves many of the quasi-algebraic problems in what he regards as two different ways. One of these receives no special label and can thus be identified as a standard method, the method habitually belonging with the tradition of mensuration algebra as he knew it. The other is *al-jabr* (*aliabra* in Gherardo's translation). A literal translation of Nºs 3, 4 and 6 will serve as illustration:

3. And if he [a "somebody" presented in Nº 1] has said to you: I have aggregated the side and the area, and what resulted was 110. How much is then each side?

   The working in this will be that you take the half of the side as the half and multiply it by itself, and one fourth results; this then add to 110, and it will be $110^1/_4$, whose root you then take, which is $10^1/_2$, from which you subtract the half, and 10 remain which is the side. Understand!

   There is also another way for this according to *al-jabr*, which is that you posit the side as *a thing* and multiply it by itself, and what results will be *the treasure* which will be the area. This you thus add to the side according to what you have posited, and what results will be a treasure and a thing which equal 110. Do thus what you were told above in *al-jabr*, which is that you halve the thing and multiply it by itself, and what results you add to 110, and you take the root of the sum, and subtract from it the half of the root. Actually, what remains will be the side.

4. And if he has said: I have aggregated its four sides and its area, and what resulted was 140, then how much is each side?

   The working in this will be that you halve the sides which will be two, thus multiply this by itself and 4 results, which you add to 1<40 and what results will be 1>44, whose root you take which is 12, from which you subtract the half of 4, what thus remains is the side which is 10.

. . . .

6. And if he has said: I subtracted its sides from its area and 60 have remained, how much thus is each side?

   In this the working will be that you halve the sides which will be two. This you thus multiply by itself and add it to 60, and take the root of the sum which is 8, to this you thus add half the number of sides, and what results will be 10 which is the side.

   But its working according *al-jabr* is that you posit the side as *a thing*, which you multiply by itself, and *a treasure* results which is the area. From this then subtract its four sides, which are *4 things*; thus remains *a treasure* minus *4 things* which equals 60, restore thus and oppose, that is that you restore the treasure by the 4 things that were subtracted, and join them to 60, and you will thus have a treasure which equals 4 things and 4 dragmas. Do thus what you were told above in the sixth question [of *al-jabr*], that is that you halve the roots and multiply them by themselves and join them to the number and take its root, and what results will be that which is 8. To this you then join the half and 10 results, which will be the side.

This piece of text calls for a number of commentaries. First we observe that the numerical steps of the basic and the *al-jabr* methods coincide (which is actually noticed by Abū Bakr, as can be seen by his identification "that which is 8" in N^{o} 6). The difference between the two methods must thus depend on something else (even though, in certain other problems, the two also differ numerically).

Al-jabr is evidently the technique explained by al-Khwārizmī, and Abū Bakr's treatise on mensuration must have been produced as a companion piece to an explanation of *al-jabr*—though not to al-Khwārizmī's treatise but to something in more archaic style. This appears from certain terminological peculiarities: more precisely from the use of the terms "restoration" (Arabic *al-jabr*) and "opposition" (Arabic *al-muqābalah*), precisely the ones that had given the technique its name.

Al-Khwārizmī uses "restoration" exclusively about the elimination of a subtractive term, in the way it is employed in Abū Bakr's N^{o} 6; the elimination of a coefficient by division is termed differently, without distinction between coefficients larger than and smaller than 1.[14] In Abū Bakr's *al-jabr* expositions, "a treasure minus 4 things" is "restored" as "one treasure" by the addition of 4 things, and "one fourth of a treasure" is "restored" through the multiplication by 4 (in N^{o} 55). In Abū Bakr's usage (which is confirmed in the standard treatment of N^{o} 4, and again in the genuine geometrical part of the treatise, in N^{os} 67, 100, and 102), restoration thus repairs any deficiency, whether subtractive or partitive (on one occasion it even repairs an excess by subtracting it, *viz* in Nº 55).

"Opposition" as used by al-Khwārizmī is the converse of his restoration, the subtraction of an addend on both sides of an equation. In the *Liber mensurationum*, the meaning once again is less specific and mostly different. Where al-Khwārizmī has the recurrent phrase "restore, and add" (the restoration being the elimination of a subtractive term $-t$ on one side of the equation, and the addition the concomitant addition of an additive term t on the other), Abū Bakr has "restore, and oppose" (Nºs 5, 6, 9, etc.);[15] in one place (Nº 22), the term covers an al-Khwārizmīan opposition; and repeatedly, when an entity A is "opposed with" or "by" another entity B, the meaning is that the equation $A = B$ is formed (most clearly in Nºs 41, 48, 49 and 50, but also in Nºs 7, 24, 25, 31 and elsewhere). Summed up in *one* concept, "opposition" means "putting on the opposite side," either in an already existing equation or by establishing an equation.[16]

Abū Bakr is not alone in not complying with the usage which was canonized thanks to the fame of al-Khwārizmī's treatise. Even al-Karajī, though he *defines* the terms as does al-Khwārizmī, uses "opposition" in Abū Bakr's way.[17] There can be little doubt that Abū Bakr's loose parlance is original and al-Khwārizmī's stricter usage an innovation, in all probability an intentional and premeditated innovation: the natural trend for the terminology of a mathematical culture undergoing a process of dynamic maturation (as that of ninth to tenth-century Islam) is to increase its precision and stringency, not to abandon its accuracy. Abū Bakr's *al-jabr* is thus pre-al-Khwārizmīan, if not *necessarily* by date then at least in substance and style (but given the triumph of al-Khwārizmī's *Algebra* it cannot then be too much later).

So much concerning the *al-jabr* method. Returning to the standard method we remember that it did not (or did not always) differ from *al-jabr* in its numerical steps. None the less it was regarded as something different by Abū Bakr. Why?

A first observation to make is the care with which the *al-jabr* sections explain that the treasure represents the area of the square, and the root (or "the thing," which is used in the same sense until standard equations are derived)[18] its side. The implication is that treasure and root/thing are *not* in themselves understood geometrically but as numbers. The basic method may then differ from *al-jabr* precisely by referring directly to the geometric method.

This conjecture is confirmed by several further observations. One concerns the word "understand" (*intellige* in the Latin text), whose occurrences are scattered throughout the work, in somewhat varying contexts. On two occasions, the word stands as an exhortation to penetrate a deliberately opaque and superfluously intricate computation and to grasp why it works after all (Nºs 50 and 74). In a number of questions concerned with genuine geometrical computation it asks the disciple to look at or understand from actually appearing diagrams why the computation is correct (a square with diagonal in Nº 2; an isosceles trapezium in Nº 78; etc.); this recalls another Gherardian translation from an Arabic text, according to which the Indians "possess no demonstration [for a particular construction] but only the device *intellige ergo*"—where indeed Indian geometrical texts have the phrase *nyāsa*, "one draws" (etc.) followed by a diagram when they want to illustrate a rule, algorithm or algebraic identity which has just been stated.[19] Finally, the word is used repeatedly as in Nº 3, that is, after the presentation the standard solution (but not the *al-jabr* solution) of a quasi-algebraic problem. Even though no diagrams are given on these occasions i Gherardo's version, the parallel to the real geometric problems suggest that here too the exhortation may have referred originally to understanding through a diagram—in Nº 3 to a diagram similar to Figure 2.

Significantly, some of the solutions which carry the "understand" are termed in a way which shows that the original constitutive geometrical entities are thought of all the way through. One instance is Nº 43, dealing with a rectangle (a "quadrate longer on one side") and indeed a rectangular version of "the four sides and the area":

> If indeed he has said to you: I have aggregated its four sides and the area, and what resulted was 76; and one side exceeds the other by two. How much thus is each side?
>
> The way to find this will be that you multiply the increase of one side over the other, always [that is, whatever the actual excess] by 2, and what results will be 4. Therefore subtract this from 76, and 72 will remain. Next aggregate the number of sides of the quadrate, which is 4, and join it to the increase of one side over the other, and what results will be 6. Thus take its half, which is 3, and multiply this by itself, and 9 results, which you join to the 72, and 81 results. Then take its root, which is 9, and subtract from it the half of 6, which is 3, and the shorter side will remain, which is 6. To this then add 2, and the longer side will be 8. Understand.
>
> The way according to *al-jabr*, however, ...

The numerical steps can be explained in several ways; algebraically, we may call the width z, and the length thus $z + 2$; proceeding mechanically from here we get Abū Bakr's *al-jabr* procedure. Or we may call the two sides x and y ($x = y + 2$), and observe that the area plus the sides is then $x\,y + 2x + 2y = x\,y + 4y + 2{\cdot}2 = (x{+}4){\cdot}y + 4$; if $X = x + 4$, we therefore have $X\,y = 76 - 4 = 72$, $X = y + (2{+}4) = y + 6$. The problem has thus been reduced to finding the sides of a rectangle whose area is $76 - 4 = 72$ (4 being *2 the excess* times *invariably 2*), and whose length exceeds the width by 2 + 4 (4 being the number of sides). This interpretation makes sense not only of the numbers but also of most of the words of the text—including the use of the identity-conserving "joining" of 4 to the excess, since the result is still an excess (as the Old Babylonian texts, Abū Bakr distinguishes between additions, even if less sharply).

Still, some formulations remain unexplained, and x's and y's are anyhow anachronistic. The second interpretation therefore has to be reinterpreted itself in order to become relevant. This is done in Figure 5: Initially, the sides are thought of as provided with the standard width 1 (the "projection" of our Old Babylonian texts).[20] The excesses are cut off, after which the sides are "aggregated," and collectively "joined" to the excess. The rest goes as in Figure 2: The excess of the rectangle over the square is bisected and a gnomon is formed, to which the quadratic complement is "joined," etc.

That the text refers to something more than mere numbers is confirmed by the recurrent phrase "what results/remains will be ...". The *al-jabr* sections (where we have the advantage of knowing what goes on) demonstrate that the phrase is no mere stylistic whim. Here the phrase also turns up time and again—but never in places where "what results" is nothing but the outcome of a computation. Instead of "what remains will be 72," such passages simply tell that "72 results". Invariably, "what results" is either a composite algebraic expression or equation, or a *something* which is identified with *something different*—as in the end of N° 3, where the numerical outcome of the algorithm is told to be the side, and again toward the end of N° 6.

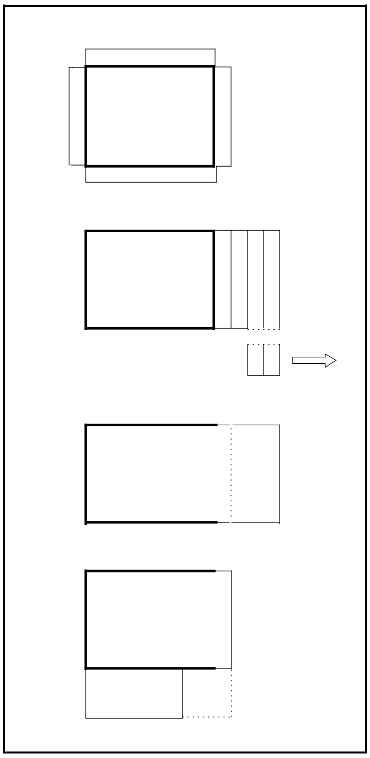

Figure 5. *Liber mensurationum*, the procedure of N° 43.

Even within the descriptions of the standard method, we therefore have to read the phrase "what results will be *a*" as "the thing which results will have the numerical value *a*". But since it is never explained, as done in the *al-jabr* sections, that something different represents the geometrical entities that the problems deal with, then the "things" whose existence is presupposed must *be* geometrical entities, derived by means of geometrical operations from the entities referred to in the statement. In N° 43, "the thing that is 4" will hence be the piece which is removed from the two rectangles representing the lengths—that is, the small square that is eliminated in the second step in Figure 5; and "the thing that is 6" will be the excess of the new length over the width.

N° 38—a kind of rectangular counterpart of N° 1—may be even more elucidating, because the solution builds on a fallacy which turns out to make excellent sense in a diagram:

> If indeed he has said to you: I have aggregated its longer and shorter sides and the area, and what resulted was 62; and the longer side exceeds the shorter by two. How much, then, is each side?
>
> The way to find this will be that you subtract 2 from 62, and 60 remains, then add 2 to the half of the number of sides, and 4 results. Join this to 60, and 64 results. Thus take its root, which is 8. This, in fact, is the longer side. And if you want the shorter, subtract 2 from 8, and 6 remains, which is the shorter side.

Figure 6 shows what goes on: We start as before, but this time, taking advantage of the coincidence between the number of sides involved and the excess (and thereby depriving the solution of any general validity), we produce the gnomon by moving the width to a position along the length and splitting off the excess from the length. The gnomon is completed as a square by fitting in the loose end of the length together with another piece (with width 1 and length) equal to “the half of the number of sides” (that is, equal to the number of sides actually involved). The area of the completed square being 64, its side (which equals the length according to the diagram) is 8.

The correct solution of N° 43 might in principle have been obtained by other means than the use of a diagram (there are always many ways to obtain a correct result), even though it seems difficult to explain the precise phrasing without the geometrical cut-and-paste interpretation. The lapses of N° 38, on the other hand, can only have resulted meaningfully from a representation where it goes without saying, firstly that the excess of length over width equals the number of sides involved, and secondly that the two together contain the completing square (the number of sides translated into “projections”)—that is, in a geometrical representation drawn or imagined in more or less correct proportions. All in all we may confidently conclude that Abū Bakr’s standard method was based on geometrical operations—and that at least the method used in the problems translated above was in naive cut-and-paste style.[21] Moreover, the geometrical operations concern the very entities which define the problems[22]—and these, as pointed out in passing above, are always *geometrically meaningful*. They do not involve entities like αQ or βs (or $\gamma l_1 - \delta l_2$) but instead: the single area; the side, both sides, or all four sides; the two diagonals of a rhombus; etc.

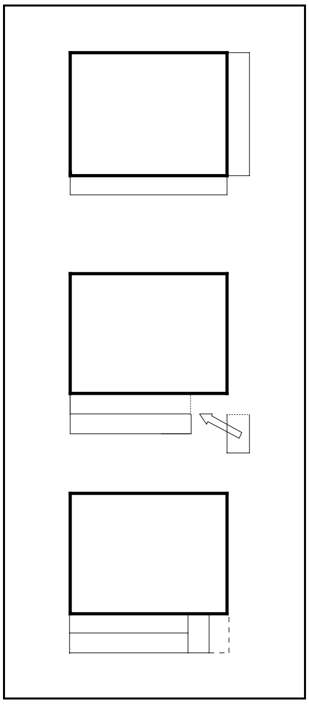

Figure 6. *Liber mensurationum*, the procedure of N° 38.

The geometrical technique of Abū Bakr’s mensuration algebra recalls what one encounters in Old Babylonian texts, and “the four sides and the area” certainly recalls BM 13901, N° 23. No surviving Babylonian problem possesses precisely the structure of Abū Bakr’s N^{os} 38 and 43, but one text (also belonging to the early phase of the development of Old Babylonian algebra) contains a close parallel, which happens also to make use of a trick for its solution which corresponds to a change of variable: AO 8862 N° 1.[23] Here, in symbolic translation, $x\,y + (y–x) = 3`3°$, $y + x = 27$; by addition, $x\,y + 2y = (x+2)\cdot y = 3`30°$ or $X\,y = 3`30°$, $y + X = 27 + 2 = 29$.

Several other similarities between the Old Babylonian corpus and the standard part of Abū Bakr’s quasi-algebraic problems can be enumerated: in particular, certain shared characteristic methods; furthermore, a highly systematic and rather intricate shift between past and present tense and between the first, second, and third grammatical person (there is also one significant though only partial divergence in this domain, which we shall discuss below). We may thus safely conclude that the two kinds of quasi-algebra are somehow connected. *How* they are connected is a question to which we shall return.

IV. Twelfth- and thirteenth-century evidence

First, however, we shall look at two later authors who still draw on the same tradition: Abraham Bar Ḥiyya—better known as Savasorda, from a twisted pronunciation of his court title—and Leonardo Fibonacci.

Savasorda's early twelfth-century *Hibbur ha-mešihah we'tišboret* (*Collection on Mensuration and Partition*), translated into Latin by Plato of Tivoli as *Liber embadorum* (*Book of Areas*)[24], has its main emphasis on genuine geometrical computation, in clear contrast to Abū Bakr's work. Equally in contrast to Abū Bakr, Savasorda also draws on the *Elements*, first in the initial chapter, where he copies the definitions from *Elements* I and VII and a number of theorems, and later in the work in a number of proofs. At one point (chapter 2, part 1, §7), however, he tells that before going on with triangles and with those quadrangles whose treatment presuppose triangulation, he will present some problems "so that by solving them, with God's assistance you may prove yourself a keen and swift enquirer". First come some problems concerning squares:

§8. $s = 10$, d?
§9. $d = \sqrt{200}$; s?
§10. $Q - {}_4s = 21$, Q? s?
§11. $Q + {}_4s = 77$, Q? s?
§12. ${}_4s - Q = 3$, s_u? (Both solutions are given).

Without doubt Savasorda has borrowed this sequence of problems, and no doubt it is related to what we encountered in the *Liber mensurationum*. It is uncertain, however, and rather implausible that he used Abū Bakr's manual directly. If he had done so and then made the present meagre selection, changing furthermore the order in §§9–11 and the value of the unknown in §§10–11, it does not seem likely that he would keep §12 unchanged (comparison between the treatments of rectangles in the two treatises supports this conclusion). That the side of §§10–11 is precisely 7 is also in itself noteworthy, as possibly related to the crude approximation that was behind Abū Bakr's N[os] 16 and 18 (side 10 and diagonal 14).

Abū Bakr's standard method appeared to be a geometrical cut-and-paste procedure referring to geometrical diagrams, but at least Gherardo's translation brings no diagrams beyond those that show the square, the rectangle, the rhombus (etc.) with which the problems deal. Savasorda's manual does contain diagrams demonstrating the correctness of his solutions (on the other hand, Savasorda provides no *al-jabr* solutions).[25] Formally, however, these refer to the Euclidean theorems which are reported in the introduction. It is therefore possible that they have been associated afresh with the traditional problems by some editor (Savasorda or a predecessor) with Euclidean schooling or familiar with Thābit ibn Qurrah's *Verification of the Problems of Algebra through Geometrical Demonstrations* (which proves the correctness of the standard algorithms of "the *al-jabr* people" for the solution of mixed second-degree problems by means of *Elements* II.5–6 in a way which is very similar to Savasorda's). It could also be, however, that this editor simply reformulated a number of traditional and still current naive geometrical procedures in Euclidean style—this would be quite easy, since the Euclidean theorems in question look precisely as "critical" recastings of a naive cut-and-paste inheritance (compare, for instance, *Elements* II.6 with Figure 2; the argument is specified below, see p. 22): in other words, it is possible but not sure that Savasorda's diagrams descend directly from the procedures traditionally connected with his quasi-algebraic problems.[26]

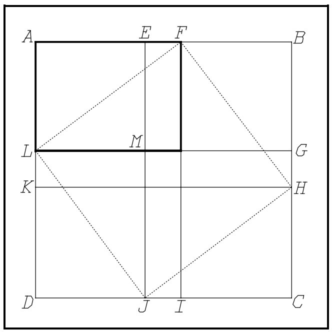

Figure 7. The naive diagram showing that $d^2 \pm 2A = (l_1 \pm l_2)^2$ in a rectangle.

Leonardo Fibonacci wrote his *Pratica geometrie* (see note 4) in 1220, and certainly drew on many sources. As Maximilian Curtze pointed out in the critical notes to the *Liber embadorum*, Savasorda is one of them. The whole structure of the work indicates that Leonardo has read the *Liber embadorum*. Quite a few of the shared features, however, derive not from direct borrowing but from one or more shared sources.

This regards precisely the group of problems which concerns us here. As pointed out by Curtze, Savasorda's §§8–12 recur in the *Pratica*. Their order, however, has been changed, as has some of the parameters (+*n* counts lines from the top, –*n* from the bottom of the page).

p. 58^{+6}. $s = 10$, d?
p. 58^{-3}. $d = \sqrt{200}$; s?
p. 59^{+5}. $Q + {}_4s = 140$, Q? s?
p. 59^{-15}. $Q - {}_4s = 77$, Q? s?
p. 60^{+10}. ${}_4s - Q = 3$, s_u? (Both solutions are given).

The formulations, furthermore, are wholly different from Savasorda's, even though at other places (for example, when Abū Bakr's N° 38 is reproduced—cf. below) the phrases of a source are taken over without any change beyond grammatical polishing. Most decisive, however, is that several of Leonardo's deviations from Savasorda agree with the "background tradition" as we know it from Abū Bakr. Like the latter in Gherardo's translation, Leonardo refers to *quatuor eius latera*, while

Savasorda takes away *omnium suorum laterum in unam summan collectum*; and like Abū Bakr, Leonardo's side in the problem $Q + {}_4s = A$ is 10.[27]

There can be no doubt that Leonardo had Gherardo's version of the *Liber mensurationum* (in full or in excerpt) on his desk while writing parts of the *Pratica*. A striking proof is provided by the problem dealt with from p. 66^{-13} onward, which coincides with Abū Bakr's N° 38 (see above, p. 12):[28]

> Again, the two sides with the expanse amount to 62; and the larger side exceeds the smaller by two. How much then is each single side?
>
> The way to find this will be that you subtract 2 from 62, and 60 remain, then add 2 to the half of the sides, and 4 result. Join this to 60, and 64 result. Thus take their root, which is 8. That, in fact, is the longer side. And if you want the shorter, subtract 2 from 8, and 6 remain, that is the shorter side. For example: posit the smaller side as a thing, then the larger will be a thing and two dragmas. From the multiplication of this shorter side by the longer results the expanse. Therefore multiply the thing, that is the smaller side, by the thing and by two dragmas, and you will have a treasure and two roots as the expanse; which, if you add to them the two sides, namely 2 roots and 2 dragmas, will be a treasure and 4 roots and 2 dragmas, which equal 62 dragmas. Remove 2 dragmas in each place, and a treasure and 4 roots remain, which equal 60, and so on.

We see that the statement differs from Abū Bakr's—among other things, Leonardo speaks here about the "larger" and "smaller" side, where Abū Bakr/Gherardo has "longer" and "shorter". In the end, Leonardo gives a solution by means of *al-jabr* (which he seems to regard as an explanation, even though completion of the *al-jabr* procedure would highlight the fallacy),[29] where Abū Bakr has none in this particular problem. In the description of the standard procedure, however, all he has done is to change the grammatical number, considering "60" etc. as plurals and not singulars.

In other places, Leonardo has geometrical proofs, some of them similar to those of Savasorda. We may look at Leonardo's treatment of "the four sides and the area" (p. 59^{+5}):

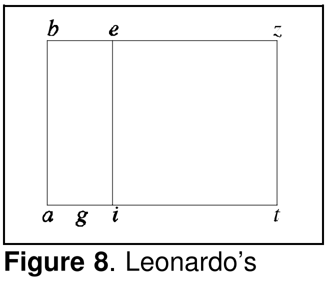

Figure 8. Leonardo's diagram for "the area and its four sides make 140".

> And if the surface and its four sides make 140, and you want to separate the sides from the surface. Let a quadrate *ezit* be put together, and the rectangular surface *ae* added to it. And let *ai* prolong the straight line *it*, and *be* prolong the straight line *ez*; and let each of the straight lines *be* and *ai* be 4 because of the number of the sides of the quadrate; because the surface *ae* equals four sides of the quadrate *et*, since the side *ei* of the latter is one of the sides of the surface *ae*; and the surface *et* contains indeed the expanse of the quadrate *zi*, and [not] its four sides. Therefore the surface *za* is 140; and that is what we have said, namely that the treasure with four roots equal 140; and the treasure is the quadrate *et*, and its four roots are the surface *ae*. Divide indeed the straight line *ai* in two equals at the point *g*; and because the line *ti* is added to the line *ai*, then the rectangular surface *it* on *at* with the square on the line *gi* will be equal to the quadrate on the line *gt*. But the surface *it* on *at* is as the surface *zt* on *at*, since *it* is equal to *tz*. Thus the surface *zt* on *at* with the square on the line *gi* equals the square on the line *gt*. But *zt* on *at* is the surface *za*, which is 140. Which, when the square on the line *gi*, namely 4, is added to them, give 144 as the quadrate on the line *gt*; therefore *gt* is 12, namely the root of 144. Therefore, if *gi*, namely 2, are dropped from *gt*, remains *it* as 10, which is the side of the quadrate *et*; whose expanse, namely 100, if its four sides are added, which are 40, will be 140,

> as claimed. And like this is done in all questions in which a number equals one square and roots, namely that to this number is added the square of the half of the roots, and the root of the sum is found; from which the half of the posited roots is removed, and the root of the treasure which is asked for will remain; which when multiplied by itself makes the treasure. For example: 133 dragmas equal one treasure and twelve roots. Therefore, if we add the square on the half of the roots, namely 36, to 133, they will make 169; when 6, namely the half of the roots, is subtracted from its root, namely from 13, 7 will remain as root of the treasure asked for; and the treasure will be 49.

The geometrical proof is similar to Savasorda's (and Thābit's), and the same observations could be made. The treatment of the problem "the two sides with the expanse amount to 62 ..." (above, p. 16) supports the conclusion that Leonardo has no direct access to the naive procedures which had still been known to al-Khwārizmī and Abū Bakr. It is also characteristic that Leonardo only gives an *al-jabr* treatment of the "four sides and rectangular area" (Abū Bakr's Nº 43, where the naive procedures were most clearly reflected in the phrasing—see above, p. 11).

This would go by itself if Leonardo's only windows on the tradition were Savasorda and Abū Bakr/Gherardo. Plausibly, however, he has also known at least one other version of Abū Bakr's manual or a close relative of this work. Gherardo, indeed, had worked on a defective manuscript, as revealed by certain corrupt passages and by references backward to problems which in the actual manuscript come later. Among the seemingly corrupt passages is the solution of problem Nº 14, "I have aggregated the four sides [of a square], and they are $^3/_8$ of its area." At the corresponding place, Leonardo has "the four sides and $^3/_8$ of the expanse equal $77^1/_2$". It is unlikely that Leonardo (who was a fairly systematic writer) should have produced this problem in order to repair the defect in Gherardo's version, since the problem is preceded by ${}_4s = {}^2/_9Q$, and followed by ${}_4s = Q$ and ${}_4s = 2Q$. It is also remarkable that Leonardo this time mentions the sides before the area, as done by Abū Bakr and in our Old Babylonian tablet. In the preceding treatment of the problem "sides plus area equal 140," Leonardo has indeed normalized the order of the members; there is certainly no reason to expect that he would innovate in this respect when repeating an inherited problem and return to the ancestral idiom when inserting a problem of his own making. The problem will hence have been borrowed, if not from a different version of the *Liber mensurationum*, then from its closest kin.

Savasorda, Gherardo and Leonardo have thus been in touch with at least three different versions of the quasi-algebraic tradition to which the problem of "the four sides and the area" belongs (as we shall see below, Pacioli seems to use material stemming from a fourth version). All these versions, however, appear to have lost contact with the original naive-geometric techniques, replacing (or possibly recasting) those proofs which allowed that with corresponding propositions from *Elements* II, and handing down those solutions which did not allow such Euclidization (like

Abū Bakr's N^{os} 38 and 43) without geometrical support (which explains why Leonardo gave up in front of N^{o} 38, cf. above, note 29).[30]

The transformation of the tradition between Abū Bakr's and Leonardo's time, and its gradual assimilation to an increasingly geometrized *al-jabr* tradition, is also shown by another feature. Abū Bakr, as we remember, took great care to distinguish the "standard procedure" from the *al-jabr* method, and to explain how "the treasure" of the latter represented the area of the square (etc.). Savasorda, as we saw, was even more respectful of the geometrical tradition, and does not mention the *al-jabr* tradition (which would anyhow, one may presume, not have been be very informative for his intended public); his only algebraic theory is borrowed from *Elements* II. Leonardo, as we see, and as it is made even more explicit in the beginning of the section on quadrilaterals (pp. 56f), has abolished the distinction completely. Where al-Khwārizmī tells number to fall into three classes, *roots*, *treasures*, and simple numbers without any reference to either[31], Leonardo tells the three natures of numbers and their fractions to be *roots of squares*; *squares*; and simple numbers: this in spite of obvious al-Khwārizmīan inspiration for the passage in question (revealed by characteristic phrases borrowed from Gherardo's translation of al-Khwārizmī).

Savasorda's and Leonardo's texts thus tell us two things. Firstly, that the tradition carrying the problem about "the four sides and the area" was still present in their world. Secondly, that it had been reduced to a shadow; after having served al-Khwārizmī's coordination of *al-jabr* with geometry, and after centuries of coexistence with the Euclidization of applied geometry, it had no mathematical standing of its own, and it only survived as a collection of venerated problems. As Gherardo must somehow have tried to express when translating Abū Bakr's *al-jabr* as *aliabra*, *algebra* had come to encompass much more than the purely numerical technique of the pre–al-Khwārizmīan *al-jabr* people.

V. Reconstructing the process

In the closing section we shall consider the end of the disintegration process. Since, however, the forces at work in this phase differ from those which shaped the earlier development, it may be convenient to discuss first what we can learn about the prehistory of algebra by following the career of "the four sides and the area" and its cognates from the cradle through the High Middle Ages. This we shall do, on one hand by summing up and connecting observations which were already made above, on the other by drawing new conclusions.

The first question concerns precisely the cradle. Our earliest encounter with the tradition and the characteristic problem embodying it was in an odd corner of an Old Babylonian mathematical scribe school text. Several features of the formulation of the problem, however, hinted at real surveying practice—and our next encounter with the problem was in an Islamic handbook concerned with that very practice. Is it likely that a problem created within the tradition of scribe school algebra but dressed as a real problem for surveyors would be adopted by these together with a narrow selection of other problems and continued as a tradition of mensuration algebra, while the main body of Old Babylonian algebra would remain the exclusive property of the scribe school and die with it? Or should we rather expect the scholar-scribes to be the debtors?

The question is a variant of a traditional problem of folklorists: Are folktales *gesunkenes Kulturgut*, as the Romanticists believed, or not? Are folktales the remnants of myths and high-level literature, or are myths created on the basis of folk tale motifs? In the final instance: Is genuine culture produced by prophets, priests and scholars alone, and the low culture of other strata merely derivative, misconstrued, and defective?

Several observations speak decisively against the hypothesis of a scribe school descent, and in favour of an origin of the mensuration algebra among practical geometers. One of these is the length of the side of the Old Babylonian version of "the four sides and the area". As in Abū Bakr's and Leonardo's corresponding problem, it is ten—but *ten minutes*. Now, 10 is an obvious choice in any culture using a decadic number system; 10´, however, is not—neither *a priori* nor according to the Old Babylonian tablets. Indeed, 10 in any order of sexagesimal magnitude (including 10°) would be an untypical side length in any Old Babylonian text. It is highly improbable (to say the least) that the queer problem should have been invented within the scribe school and been constructed around the anomalous value of the unknown side, and then taken over by people who by accident could correct 10´ (which they would see as $^1/_6$) into the obvious value 10. The scribe school mathematician, however, if borrowing a problem with the parameter 10, could reasonably be expected to locate this number in his habitual order of magnitude, which in the tablet in question is that of minutes.

Another observation has to do with the topic and general character of the problem. As already hinted at, the combination of the geometrically meaningful (*all four* sides of a square field) with the practically meaningless (which practitioner ever knew the sum of the sides and the area without first knowing them separately?) gives the problem the character of a bizarre *riddle*. Such riddles, when mathematical, are known as recreational problems. In pre-Modern times, they were transmitted within environments of mathematical practitioners, where they served the purpose told by Savasorda: "that by solving them, with God's assistance you may prove yourself a keen and swift enquirer"; or, in another formulation taken from a Carolingian problem collection (I quote the puzzle in full):

> A paterfamilias had a distance from one house of his to another of 30 leagues, and a camel which was to carry from one of the houses to the other 90 measures of grain in three turns. For each league, the camel would always eat 1 measure. *Tell me, whoever is worth anything*, how many measures were left.[32]

In other words, these problems—which according to their dress belong within the domain of the practitioners in question (surveyors and caravan traders, respectively) but which are more complex or more bizarre than the problems solved in everyday practice—serve to train the mental agility and enhance the professional self-esteem of the members of the craft (whence the term "brilliant" used by al-Khwārizmī to characterize the useless second-degree part of *al-jabr*—cf. above, p. 6).[33] Invariably, they have something stunning in their formulation: unless a clever trick is applied (an intermediate stop), the camel will eat *exactly* everything; in another widespread problem, 100 monetary units will buy *exactly* 100 animals; repeated doublings run to 30 or 64, because this fits the days of the month or the cases of a board game; etc.[34]

The topic—the real sides of a real field; the striking parameter—exactly all four sides; and the solution by means of a doubly weird trick—quadripartition and quadratic completion: all three features indicate that "the four sides and the area" was hatched not in a scribe school but in a non-scholastic environment of practical geometers.

A third observation allows us to locate this environment tentatively in time and space. As stated above (p. 13), Abū Bakr's discourse is astonishingly close to what we find in Old Babylonian school texts. There is one exception to this rule, however. Abū Bakr always has a hypothetical "somebody" posing the question (in the first person singular, past tense). Old Babylonian texts, instead, start directly with the question (as in BM 13901, Nº 23), implying that it is the teacher who asks. One group of texts, however, starts its problems with the familiar "if somebody has asked ...". These texts come from Tell Harmal and Tell Dhibaʾi, both in the Kingdom of Ešnunna, and belong to the earliest eighteenth century B.C.[35] Ešnunna is an early focus for that Akkadian scribal culture which arose around the mid-Old Babylonian period: late nineteenth century Ešnunna produced the first law code in Akkadian, half a century in advance of the Codex Hammurapi. Since algebra is an Akkadian genre with no identified Sumerian antecedent, Ešnunna may thus be the location where the recreational lore of Akkadian-speaking practical geometers was adopted into the curriculum of the Akkadian scribal school.

An Akkadian origin fits the side of our square field. Akkadian, as Arabic (and as the likely intermediate carrier language of our tradition, Aramaic), is a Semitic language and has a decadic number system. It also fits the name "Akkadian method" given to the quadratic completion in a late Old Babylonian mathematical text; it agrees with the observation made by Robert Whiting that the problems contained in a school text from the Old Akkadian period (the 22nd century BC) dealing with area measurement are so much facilitated by familiarity with the geometric-"algebraic" rule $(R–r)^2 = R^2 – 2Rr + r^2$ that this rule is likely to have been presupposed; and it matches the presence of a tablet with a bisected trapezium (another favourite problem following our tradition until Abū Bakr and Leonardo) in an Old Akkadian temple.[36] It looks as if already the Old Akkadian scribe school had adopted part of the recreational lore of the Akkadian surveyors, but that the strictly utilitarian neo-Sumerian school (21st century BC) did not transmit it.[37]

Since there is, anyhow, close affinity between the Old Babylonian scribe school algebra and the tradition of mensuration algebra, it is reasonable to assume the former to have developed from the adoption of the latter under the fecundating influence of the systematic spirit of the school. The quadratic completion, originally another weird trick comparable to the quadripartition and the intermediate stop, may have been the cornerstone on which the whole stupendous edifice of Old Babylonian algebra was erected.

The overlap between the algebra of the scribe school and that of the *Liber mensurationum* (and other post-Babylonian sources) allows us to draw up a list of problems which can be ascribed with some confidence to the mensuration algebra of the early Old Babylonian epoch. Of course (sticking to the symbols introduced on p. 8), $s + Q = \alpha$ and ${}_4s + Q = \beta$ (we may even be confident that $\alpha = 110$, $\beta = 140$); probably also problems with differences (area minus side(s), and side(s) minus area) and questions about the diagonal when the side is given, and vice versa. For rectangles, furthermore, $A = \alpha$, $l_1 \pm l_2 = \beta$; $A + (l_1 \pm l_2) = \alpha$, $l_1 \mp l_2 = \beta$; $A = \alpha$, $d = \beta$ (this latter problem is found on the Tell Dhiba'i-tablet). Highly likely is also the presence of problems dealing with several squares, at least $Q_1 \pm Q_2 = \alpha$, $s_1 \pm s_2 = \beta$ (a partial alternative, less plausible however, is the presence of the rectangle problems $l_1 \pm l_2 = \alpha$, $d = \beta$).[38] Rhombi and right triangles (both of which are used as pretexts for the formulation of quasi-algebraic problems in the *Liber mensurationum*) seem to be beyond the horizon, as is anything involving non-right triangles.

Old Babylonian scribal algebra developed into a sophisticated discipline, but most of its higher achievements were lost when the Old Babylonian era was interrupted by conquest and social breakdown after 1600 BC, at which occasion the scribe school also disappeared. The late Babylonian period, in particular in the Seleucid era (from 300 BC onwards), produced a certain revival of algebraic activity, it is true; discontinuity in the use of Sumerian word signs demonstrate, however, that much the transmission had taken place outside the scribal environment, and that a readoption of material from the mensuration algebra tradition occurred.

In the meantime, it appears that new problem types had been invented or imported into this tradition. The most systematic Seleucid treatment of second-degree problems is found on the tablet BM 34568.[39] All problems except two deal with rectangles, where various combinations of sides, diagonal and area are given.[40] With a single exception, the rectangle problems recur in the *Liber mensurationum* (at times with other parameters); moreover, the exception ($l_1 + D$ and $l_2 + d$ given) is not really one, since Abū Bakr's Nº 36 ($l_1 + d$ and $l_1 - l_2$ given) is reduced to the Seleucid problem and then solved in the same way.

Interestingly, the only rectangle problem dealing with a diagonal of whose presence in the early mensuration algebra we are sure (*viz* $A = \alpha$, $d = \beta$, found in the Tell Dhiba'i tablet) is absent from the Seleucid anthology. Also interesting is one of the two problems in the tablet which do not consider rectangles. It deals with a reed leaning against a wall, and is equivalent to the rectangle problem $d - l_1 = \alpha$, $l_2 = \beta$ (Abū Bakr's Nº 31). Nothing with the same mathematical substance is found in the Old Babylonian corpus. *The dress*, on the other hand, is familiar, but originally it covered a problem translatable into the much more trivial $d = \alpha$, $l_1 = \alpha - \beta$.

On the whole, the Seleucid tablet thus looks like a listing of *new* problems; the reed problem may be meant to demonstrate how this fascinating new wine could be poured into an old cherished bottle, thereby lending new quality to both. In any case, and quite in contradiction to the traditional view, the tablet demonstrates the discontinuity of Babylonian mathematics in spite of apparent continuity.[41]

Also at variance with widespread convictions, but the other way round, is the perspective we get on the core of *Elements* II if we correlate propositions 1 to 10 of the Euclidean work with what we have come to know about mensuration algebra.[42] Postponing for a moment propositions 1 to 3, the rest can be seen as quasi-Kantian critiques of the familiar procedures: prop. 4 is used, e.g, by Leonardo when he finds the sum of the sides of a rectangle from the diagonal and the area, while Savasorda (proceeding like the Tell Dhiba'i text) finds their difference via prop. 7;[43] prop. 6 explains the solution of all problems $Q \pm \alpha s = \beta$ (including "the four sides and the square") and $A = \alpha$, $l_1 - l_2 = \beta$ (and Leonardo quotes it on these occasions); prop. 5 has a similar relation to rectangular problems $A = \alpha$, $l_1 + l_2 = \beta$ and to $\alpha s - Q = \beta$ (again noticed by Leonardo); prop. 7, beyond the use made of it by Savasorda, explains the rule which seemed to be presupposed already in an Old Akkadian school text (cf. above, p. 21); prop. 8 does not seem to enter any problem directly which we have discussed so far; but it may be connected to the configuration of "four sides and area" (showing that, if we add the four sides to a square $\square(s)$, we do not get a square $\square(s+2)$—instead, we have to add the four sides of the average square $\square(s+1)$; conversely it can be linked with the concentric inscription of one square into another (also familiar from Old Babylonian practical geometry). Propositions 9 and 10, finally, which like prop. 8 serve nowhere else in the *Elements* (and which must therefore have been supposed to possess a value of their own),[44] solve the problems where the sum of two square areas and either the sum or the difference between their sides are known[45] (Leonardo also makes appeal to prop. 10 a couple of times).

The proofs of propositions 9 and 10 are obviously of the Greek and not the naive type. The others, however, fall into two sections, of which the second is in essence a cut-and-paste proof, and the first explains why the various constituents of the diagram are really squares, rectangles etc. Section 1, we may say, takes care that the subsequent cut-and-paste section is not naive.

Propositions 1 to 3 have a similar function. Prop. 1 is a general "critique of mensurational reason," justifying the cutting and pasting of rectangles; propositions 2 and 3 apply this insight to the particular situations where sides (provided with a "projection," it goes by itself) are added to or subtracted from a square.

Elements II.1–10, we may hence conclude, is closely connected to the cut-and-paste mensurational algebra and is precisely, as formulated above, *a critique*. We may observe, furthermore, that the whole group of propositions points back to the stock of problems and procedures which seems to have been present already in Old Babylonian times. There is no trace of the new problem types from the Seleucid tablet.

Arguments can be given that the kind of area geometry which was canonized in *Elements* II was developed in the fifth century BC in connection with a theoretical investigation inspired by surveyors' geometry and algebra.[46] If this is really so, then there is some reason to believe that the new problems reached or arose in the Near Eastern and Mediterranean world after 500 BC, but before 200 BC. We may think, either of the contacts resulting from Alexander's conquests, or of the general establishment of cultural interaction along the Silk Road.[47]

It may be added that the small group of second-degree problems in Diophantos's *Arithmetica* I also refer to what appears to be the original core of the mensuration algebra: a rectangle with given area and given sum of (prop. 27) or difference between (prop. 30) the sides; and two squares with given sum of the sides and given sum of (prop. 29) or difference between (prop. 29) the areas.

The next occasion on which the tradition of mensuration algebra turns up in familiar sources is at its encounter with the numerical *al-jabr* practice, and when al-Khwārizmī draws upon its cut-and-paste technique in order to demonstrate the correctness the *al-jabr* calculations. These geometrical proofs were already discussed above and need not be taken up again. Only one observation should be added: when teaching the addition and subtraction of binomials involving roots, al-Khwārizmī's standard exemplification of the root—that is, we must presume, the first square root which his reader is expected to recognize as not reducible to a number—is $\sqrt{200}$, the diagonal of our familiar 10×10-square. Unless this concurrence is purely accidental (which is not likely—cf. also note 13 on the possibility to distinguish chronological strata in the mensuration tradition by means of changing approximations to this length), the practice from which al-Khwārizmī borrowed his proofs thus appears to have been fairly well-known.

Mensuration algebra did not disappear as an independent tradition after al-Khwārizmī's integration of its methods with *al-jabr*. As we have seen, at least three or four different versions could be found in the Islamic world in the twelfth and thirteenth century. But as we have also seen, it had lost its *raison-d'être* as a separate mathematical tradition. In this as in other fields, Islamic mathematics initiated an integration of theoretical and practitioners' mathematics which was, in the Modern epoch, to transform the latter enterprise into *applied* [theoretical] *mathematics*. Gherardo, as a faithful translator, would still render Abū Bakr's sharp distinction between (geometrical) standard method and (numerical) *al-jabr*. Leonardo the mathematician, however, did not see the point, or saw no point in doing so.

VI. The End of a Tradition

However much the tradition of mensuration algebra had become superfluous from a theoretical point of view, it did not die easily in Christian Europe once it had been adopted. Thus, in the geometrical part of his *Summa de arithmetica*, Luca Pacioli tells that

> even though rather much has been said about the rule of algebra in the part on arithmetic: none the less, something must be said about it here.[48]

What needs to be said turns out to be precisely what Leonardo tells in his *Pratica geometrie*. The treatment is so close to Leonardo that misprints in Pacioli's lettering of diagrams can be corrected from Leonardo's text (this was how I stumbled upon the affinity between the texts). But there are certain puzzling exceptions to his faithfulness: Thus Leonardo, as we remember, did not speak about "the four sides and the area" but about "the area and its four sides" making up 140. Pacioli, however, returns to the original pattern. Since this pattern was as foreign to Renaissance algebra as to Old Babylonian algebra, Pacioli can not be expected to have reinvented the ancestral formula on his own: it must have been around. As it has sometimes been suspected, Italian Late Medieval algebra, however much it was indebted to Leonardo, must have received impulses from the Islamic world through supplementary channels.[49]

The last appearance of the set of problems once belonging to the tradition of mensuration algebra is in Pedro Nunez *Libro de algebra en arithmetica y geometria* from 1567 (at least the last which I know about—but my reading of Renaissance sources is far from complete). Part III, chapter 7 has the heading "About the practice of algebra in geometrical cases or examples, and firstly about squares".[50] It is obvious that Nunez has profited much from Pacioli, as also told in his concluding address to the reader (fol. 323^v). In our now customary abbreviations, the examples about squares are the following:

1. $s = 3$: Q?
2. $Q = \alpha$: s?
3. $s = 3$: d?
4. $d = 6$: s?
5. $d+s = 6$: d? s?
6. $d \cdot s = 10$: d? s?
7. $d-s = 3$: d? s?
8. $s \cdot (d-s) = 15$: s? d?
9. $d \cdot (d-s) = 14$: s? d?
10. $s+Q = 90$: s? Q?
11. $d+Q = 12$: Q? s?
12. $s+d+Q = 37$: s? d? Q?
13. $Q \cdot s = 10$: s? Q?
14. $d \cdot Q = 12$: s? Q?

These translations are misleading in so far as they conceal the real format of the examples. This format follows that of the Euclidean *Data* (and of Jordanus de Nemore's *De numeris datis*)—for instance, N° 11 tells that "if the diameter and the area of the square together are known, then each is known separately". Only afterwards the numerical example is introduced. In this respect, the text is thus developing toward *theory*. It has also dropped the opaque solutions by unexplained numerical algorithms (the rudiments of naive cut-and-paste procedures), and starts directly with the algebraic solution.

But the themes are traditional. Nunez, when advertising the capabilities of algebra, feels the need to demonstrate that this wonderful technique is able to resolve both the traditional problems and even more complex problems of the same kind (like N° 12). He only presents one example for each problem type, and thus drops "the four sides". For the last time, however, "the side" appears before the area in N° 10, betraying the Bronze Age descent—and for the last time (before Viète changed the terms in which the problem of homogeneity was discussed) it is explained that what is added to the area is another area, "a root" being the side provided with a "projection 1" (cf. also Nunez' fol. 6^r).

Within a generation, Viète was to show the capability of algebra to elucidate much more complex problems. If algebra was still in need of commercials, much more impressive applications than artificial mensuration geometry were now at hand. After somewhat more than three thousand years, "the area and the four sides," as the totality of mensuration algebra, could leave the world so quietly that nobody noticed its death, and nobody remembered that it had ever existed.

References

* Dedicated to Niels Arley.

1. Otto Neugebauer, ed., *Mathematische Keilschrift-Texte*. 3 vols. (Berlin: Julius Springer, 1935, 1935, 1937), here vol. 3, pp. 1–5. The translation is mine, as are all translations of sources into English in the following.

2. The principles of translation as well as the single operations and terms are discussed in full in pp. 45–69 of my "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought," *Altorientalische Forschungen*, 1990, *17:* 27–69, 262–354. In this work I also explain why the detailed investigation of the texts and their terminology invalidates the received interpretation of the Babylonian technique as a numerical algebra, and suggests a reading as "naive" cut-and-paste geometry (to be explained below).

3. Most of our text is written in Akkadian, the spoken language of the Old Babylonian period. Akkadian terms are transcribed in italics. The present text contains only few Sumerian terms (indicated by spaced writing), most of which are genuine loan words, and which go back to the mathematics of the Sumerian epoch (before 2000 BC). Other texts (mathematical as well as non-mathematical) may contain many more Sumerian terms, but as a rule these functioned as word signs for Akkadian speech.

4. Imagining lines as provided with an implicit standard width seems indeed to be quite common in field measurement which has not interacted (or not interacted intensely) with Euclidean abstraction. It was the practice of Ancient Egypt—see T. E. Peet, *The Rhind Mathematical Papyrus, British Museum 10057 and 10058* (London: University Press of Liverpool, 1523), p. 25; Leonardo Fibonacci describes it as the system used when land was bought and sold in thirteenth-century Pisa (*Pratica geometrie*, pp. 1–224 *in* Leonardo Pisano, *Scritti*. 2 vols., Vol. II: *Practica geometriae* et *Opusculi*, ed. B. Boncompagni (Roma: Tipografia delle Scienze Matematiche e Fisiche, 1862), p. 3f); Luca Pacioli, finally does the same for fifteenth-century Florence (*Summa de Arithmetica geometria Proportioni: et proportionalita*. (Novamente impressa. Toscolano: Paganinus de Paganino, 1523), part II, fol. 6^{v}–7^{r}).

One may ask whether the Euclidean definition of the line ("a length without width") was originally (and well before Euclid, of course) introduced with the purpose of barring this "misunderstanding" (as a Greek geometer would see the matter).

5. Those who know the terminology of Greek geometry may observe that the *dýnamis* is a square considered in the same way—cf. my "*Dýnamis*, the Babylonians, and Theaetetus 147c7—148d7," *Historia Mathematica*, 1990, *17*: 201–222.

6. It is forgotten in most general histories of mathematics but should be strongly emphasized that the place value system used in the Babylonian mathematical texts appears only to have been used for intermediate calculations (like a slide-rule, it was a pure floating point system, presupposing that the reckoner knew the order of magnitude) and in the mathematical school texts. Economical texts (of course) use other number systems where the absolute order of magnitude is fixed.

7. This is what al-Khwārizmī tells in the preface (F. Rosen, (ed., trans.), The *Algebra* of Muhammad ben Musa (London: The Oriental Translation Fund, 1831), p. 3, cf. J. Ruska's corrections, "Zur ältesten arabischen Algebra und Rechenkunst," *Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Philosophisch-historische Klasse*, Jahrgang 1917, 2. Abhandlung, p. 5). Rosen's translation was made from the manuscript Oxford, Bodleian I CMXVIII, *Hunt.* 214, fol. 1–34, as was Rozenfeld's Russian translation and the Arabic edition (al-Khwārizmī, *Kitāb al-muḫtaṣar fī ḥisāb al-jabr wa'l-muqābalah*, ed. A. M. Mušarrafa & M. M. Aḥmad (Caïro, 1939)). A close analysis of the text and comparison with Latin translations made in the twelfth century by Robert of Chester and Gherardo of Cremona shows that the text of the Oxford manuscript has been amended by at least three different editors (two of whom must antedate Robert of Chester)—see chapter V in my "'Oxford' and 'Cremona': On the Relations between two Versions of al-Khwārizmī's *Algebra*," *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1991 nr. 1. (To appear in *Proceedings of the 3rd Maghrebian Symposium on the History of Mathematics Alger, 1–3 December 1990*). For most purposes, Gherardo's version (now available in a critical edition prepared by B. B. Hughes—"Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*," *Mediaeval Studies*, 1986, *48*: 211–263) is to be preferred to the revised Arabic text; unfortunately for historians of mathematics, however, Gherardo omitted the preface as well as the second and third part of the work (the practical geometry and the arithmetic of legacies), for which we have to trust the corrected Arabic text or one of its derivatives.

8. *Verification of the Problems of Algebra through Geometrical Demonstrations*, ed., trans. P. Luckey, "Ṯābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen," *Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-physische Klasse. Berichte*, 1941, *93*: 93–114.

9. That second-degree problems constituted at least the core of *al-jabr* follows from al-Khwārizmī's introduction. Most likely, however, the formulation and solution of second-degree problems by means of "treasures," "roots" and "dirhems" (cf. below) was not only the core of *al-jabr* but also the meaning of the term *stricto sensu*.

10. An analysis of the relevant parts of this treatise, together with arguments for the dating, will be found in my "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra," *Erdem* (Ankara), 1986, *2*: 445–484; cf. also my "'Algèbre d'*al-ğabr*' et 'algèbre d'arpentage' au neuvième siècle islamique et la question de l'influence babylonienne," pp. 83–110 *in* Fr. Mawet & Ph. Talon (eds), *D'Imhotep à Copernic. Astronomie et mathématiques des origines orientales au moyen âge* (Leuven: Peeters 1992).

11. Critical edition in H. L. L. Busard, "L'algèbre au moyen âge: Le 'Liber mensurationum' d'Abû Bekr," *Journal des Savants*, Avril-Juin 1968, 65–125.

12. The text is corrupt (or possibly intentionally enigmatic, as is indeed N° 50). More or less at the corresponding place in his exposition, Leonardo (*Pratica geometrie* (cit. n. 4), p. 61) discusses the problem $_4s+{}^3/_8Q = 77^1/_2$.

13. The datum of N^os^ 16 and 18 ($d–s = 4$) points back to the crude idea that $s = 10$, $d = 14$ (the result, however, is found correctly as $4+\sqrt{32}$). In N° 19, the diameter is found as $2\cdot 7^1/_{14}$, and the problem is thus constructed backwards from the value $d = 14^1/_7$, an approximation to the length of the diagonal in a 10×10-square that is given in the beginning of the chapter. The quasi-identity between N^os^ 16 and 18 shows that the tradition has been jumbled at some point (whether during the copying of Abū Bakr's manual or in the sources on which he draws), N° 18 representing the original formulation (traditionally, differences had been told as excesses). For this reason, the "seven and one half" of N° 15 is probably to be understood as a distorted version of the "seven and one half *of a seventh*" of N° 19.

One may add that N° 12 builds on the same geometrical configuration as N° 6, a 6×10 square, and that cut-and-paste solutions of the two problems will be identical (any solvable problem "area minus four sides" stands in this relation to a problem of the type "area plus four sides"). N° 13, on its part, deals with the same square as N° 12. These two deviations from the norm may thus be understood as a cascade of derived problems. In a similar way, N° 17 may have been constructed as a parallel to N° 16 on the basis of the approximation $\sqrt{2}:1 \approx 17:12$, in which case N^{os} 16 and 17 could be related via the algorithm for side-and-diagonal numbers.

14. Cf. the following passages in Hughes' edition of Gherardo's translation (cit. n. 7): IIA:11f; IIB: 12–14; VI:18,45f,70; VII:6f,30–34,52f,84,92,119f,121f. "Opposition" occurs in VI:74 and VII:19.

15. In the passage from N° 55 where "restoration" meant the elimination of *144 dragmas* from *one treasure and 144 dragmas*, the "opposition" stands for the subtraction of 144 from the other side of the equation.

16. Speaking about "the opposite side" comes naturally when we refer to our own equations, where the sign of equation separates two sides. It comes less easily if equations are formulated in spoken words, as are al-Khwārizmī's and Abū Bakr's "rhetorical" equations. Abū Bakr's use of "opposition" thus suggests that this terminology was formed around some kind of material representation of equations (as we shall see, al-Khwārizmī's usage must be secondary), most probably a sort of scheme.

The use of schemes with opposing sides is indeed known from India—see B. Datta & A. N. Singh, *History of Hindu Mathematics. A Source Book.* (Bombay: Asia Publishing House, 1962. 1st ed. Lahore: Motilal Banarsidass, 1935–38), part II, pp. 28–32. *Al-jabr* can hardly have been borrowed from the "scientific" algebra of "scientific" mathematicians like Āryabhata and Brahmagupta, it is true—cf. my "Formation of 'Islamic Mathematics'. Sources and Conditions," *Science in Context*, 1987, *1:* 281–329, here p. 286. However, a connection to Indian practical mathematics is strongly suggested by the term "root" (Arabic *jaḏr*), used first about the square root of a number and next (via the square root of the unknown *treasure*) about the unknown of an equation. The term makes no metaphorical sense in the Arabic *al-jabr* tradition, where the root was taken of a number or of an amount of money and had no geometrical connotations (cf. below); in India, however, the square root was understood as the base of a geometrical square, and designated since early times by the term *mūla*, "base" or "root [of a tree]"—see Datta & Singh, *op. cit.*, part I, pp.169f.

17. The al-Khwārizmīan definition is found in the *Kāfī* (A. Hochheim, *Kafî fîl Hisâb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi. 3 vols. (Halle: Louis Nebert, 1878), vol III, p. 10); on the use, see G. A. Saliba, "The Meaning of al-jabr wa'l-muqābalah," *Centaurus*, 1972–73, *17:* 189–204, here pp. 199f.

18. In first-degree problems (e.g., in the inheritance algebra treated as part III of al-Khwārizmī's *Algebra*), it is customary to label the unknown "a thing"; "a root," as a matter of fact, would give no sense. One text published in Medieval Latin translation in G. Libri's *Histoire des mathématiques en Italie*, vol. I (Paris, 1838) uses "a treasure" (p. 304f), which is of course also a meaningful name for an unknown amount of money.

While the "root" may point to Indian practical mathematics, weak indications exist that "the thing" is related to Greco-Egyptian practice, either by descent or by common descent (the evidence is listed but not thoroughly discussed in my "Sub-scientific Mathematics: Undercurrents and Missing Links in the Mathematical Technology of the Hellenistic and Roman World," *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1990 nr. 3, end of chapter IV. (To appear in *Aufstieg und Niedergang der römischen Welt*, II vol. 37,3). However, since mathematical problems circulated between China and the Mediterranean no later that the first Christian century (cf. note 47), Indian and Greco-Egyptian connections are not mutually exclusive. However, since mathematical problems circulated between China and the Mediterranean no later than the first Christian century, Indian and Greco-Egyptian connections are not mutually exclusive.

19. The fragment was published by Marshall Clagett, *Archimedes in the Middle Ages.* 5 vols. Vol. V: *Quasi-Archimedean Geometry in the Thirteenth Century*, p. 599 (Philadelphia: The American Philosophical Society, 1984); references to the Indian practice will be found in my "'Algèbre d'*al-ǧabr*' et 'algèbre d'arpentage'," n. 22.

20. Even Leonardo and Nunez, when they are to explain the geometrical interpretation of *al-jabr*, refer to the "root" as a rectangle with length equal to the side of the square, and with width 1. Cf. below.

21. The conclusions do not hold for *all* problems: the voluntarily abstruse standard solution of N° 50, for instance, is a mere translation of the tortuous *al-jabr* solution which follows it; other standard solutions appear to be geometrical but do not use cut-and-paste techniques. These exceptions, however, do not concern us here, however relevant they are for a complete analysis of Abū Bakr's eclectic manual.

22. This does not go by itself even within a naive cut-and-paste algebra, as demonstrated by the Old Babylonian algebraic corpus: Old Babylonian lines and surfaces may not only represent pure numbers or prices, which permitted the scribal mathematicians to solve non-geometric problems by means of their naive-geometric technique. A line could also represent an area, which made possible the treatment of biquadratic problems (e.g., BM 13901 N° 12, which is solved as a biquadratic even though a simple quadratic solution is possible).

In contrast, a technique which restricts itself to manipulating those geometrical entities which enter the problem directly is by necessity prevented from developing into an all-purpose algebra. One might even be tempted to use this as part of a definition of *algebra* as "complex analytical computation, or theory for such computation, where intermediate steps need only have a meaning with relation to a representation but not necessarily with relation to the entities that define the problem"—in which case Old Babylonian algebra *is* algebra, but mensuration algebra is not.

23. Ed. Neugebauer, MKT I, pp. 108f (cit. n. 1). Analysis in my "Algebra and Naive Geometry," pp. 309ff (cit. n. 2).

24. An edition of the Latin text with German translation was published by Maximilian Curtze, in *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance,* pp. 1–183 (Abhandlungen zur Geschichte der mathematischen Wissenschaften, vol. 12–13. Leipzig: Teubner, 1902). In the footnotes to the edition, Curtze also traced the parallels between Savasorda's text and Leonardo's *Pratica geometrie*.

25. It is thus wholly wrong even though a generally accepted view that the treatise is "the earliest exposition of Arab algebra written in Europe" (Levey, "Abraham bar Ḥiyya ha-Nasi," *Dictionary of Scientific Biography,* vol. I, pp. 22f, quotation p. 22 (New York: Scribner, 1970)).

26. Savasorda's treatment of his §18 might be taken as an argument against his being familiar with traditional cut-and-paste procedures. Here he finds the difference between the sides of a rectangle from the area and the diagonal by means of the rule that $d^2 = 2A+(l_1-l_2)^2$, which is stated in §14 and argued there from *Elements* II.7. After that he solves the problem from the area and the difference between the sides. If he had thought of the naive diagram probably underlying his rule, however, it might also have told him that $(l_1+l_2)^2 = d^2+2A$, which would have simplified the solution (cf. Figure 7). However, an early Old Babylonian problem from Tell Dhiba'i to which we shall return (p. 20 and later) applies precisely the same method as Savasorda. Both authors (and the whole tradition) may thus have used the problem to show the combination of several standard methods.

It is noteworthy that the proof of *Elements* II.7 builds on the sub-diagram *MGCJ* of Figure 7 (without diagonals), while that of *Elements* II.4 (from which follows that $(l_1+l_2)^2 = d^2+2A$, of which Leonardo Fibonacci makes use when solving the corresponding problem) employs the complete diagram (without the lines *EJ* and *KH* and without diagonals).

27. We may also mention Leonardo's counterpart of Savasorda's §18 (cf. above, note 26), where Leonardo (like Abū Bakr) finds the sum of the sides, and refers in his proof to *Elements* II.4.

28. The two translations have been made so as to show precisely the extent and character of the agreements/disagreements between the two texts, in vocabulary as well as in the choice of grammatical forms. For the sake of creating one-to-one-correspondences, the translation "expanse" has been used for *embadum*, a term for the area which Leonardo share with Savasorda/Plato.

29. In the completion of the *al-jabr* procedure, the 4 to be added to 60 are to be found as the square on half the number of roots, not as 2 plus this half. The root (and thus the shorter side), furthermore, is found as $\sqrt{64}$ minus half the number of roots, and the longer side finally as the shorter plus 2 the difference between the sides.

All this will certainly have been recognized by Leonardo. In all probability, his "and so on" serves to conceal that he does not understand what goes on.

30. There is a vague possibility that Leonardo still had access to the habitual diagrams for a number of complex problems involving the diagonal of a rectangle (e.g., $l_1+l_2+d = 24$, $A = 48$, *Pratica geometrie* p. 68 (cit. n. 4), where he introduces diagrams which generalize the one which was shown in Figure 7. But he may also have developed these diagrams anew, since they follow without too much difficulty from the procedure.

31. Hughes, "Gerard of Cremona's Translation," p. 233 (cit. n. 7).

32. *Propositiones ad acuendos iuvenes*, problem 52, version II, ed. M. Folkerts, "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*," *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften* (Wien, 1978), 116. Band, 6. Abhandlung, here p. 74. Emphasis added.

33. This relation between professional mathematical practice and recreational mathematics is a focal theme in my "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon," *History of Science*, 1990, *28:* 63–86.

34. This characteristic has a double explanation: A riddle is always better the more surprising its formulation. Moreover, as long as the parameters of a problem are not noteworthy, they are likely to change when transmitted within a semi-oral culture; once somebody has chosen a remarkable parameter it is likely to be remembered, both because this follows from remarkability *per se*, and because it makes the riddle as a whole better.

Mathematical riddles are hence liable to be born striking, and to conserve this characteristic when they are transmitted. If by accident they are born without marked parameters, a kind of attraction law guarantees that they will acquire them soon (or that they will be forgotten).

A particular variant of the quest for the extraordinary was mentioned above: The presence in the *Liber mensurationum* of deliberately opaque and perplexing problem solutions, which the disciple is asked to look through.

35. The texts were published by Taha Baqir, in "Some More Mathematical Texts from Tell Harmal," *Sumer*, 1951, *7:* 28–45, and in "Tell Dhiba'i: New Mathematical Texts," *Sumer*, 1962, *18:* 11–14, pl. 1–3, respectively.

36. See my “Algebra and Naive Geometry,” p. 326 (cit. n. 2); R. M. Whiting, “More Evidence for Sexagesimal Calculations in the Third Millennium B.C.,” *Zeitschrift für Assyriologie und Vorderasiatische Archäologie*, 1984, *74:* 59–66, here p. 65f; and J. Friberg, “Mathematik,” *in Reallexikon der Assyriologie und Vorderasiatischen Archäologie*, vol. VII, 531–585, here p. 541 (Berlin & New York: de Gruyter, 1990).

37. Since no traces of genuine second-degree algebra are found in the Old Akkadian school texts, we may also surmise that the discovery of the quadratic completion (the “Akkadian method”) took place somewhere between the 22nd and the 19th century BC.

38. BM 13901 N^{os} 8 and 9 deal with two squares, about which the sum of the areas and the sum of/difference between the sides are stated. The square sum of the sides sides (20´ and 30´) is no square, and thus the problems cannot be transformed into rectangle-diagonal problems without a change of parameters. Evidently it is not excluded that surveyors’ rectangle-diagonal problems have been adopted and transformed, and the parameters then changed. However, reflections of our tradition in classical sources (in particular *Elements* II, cf. below) and the unquestionable presence of two-square problems where $Q_1 - Q_2$ is given speak in favour of the two-square assumption with given sum. A sequence of problems about the same two squares in the late Old Babylonian text TMS V (one of which coincides with BM 13901 N^{o} 8) speaks about the smaller square as located concentrically within the larger one—a configuration that refers to geometrical practice (E. M. Bruins & M. Rutten, eds., *Textes mathématiques de Suse*. Paris: Paul Geuthner, 1961, here pp. 46f). One of the problems (col. III, l. 4, unmentioned and untranslated in the edition) tells the difference between the areas and the difference between the sides.

39. Ed. Neugebauer, MKT III, pp. 14–17 (cit. n. 1).

40. l_1 and l_2; l_1 and d; l_1+d and l_2; l_1+l_2 and A; l_1+l_2 and d; l_1+d and l_2; l_1+d and l_2+d; l_1+l_2+d and A.

41. This discontinuity can be traced on several levels beyond those already mentioned (Sumerian word signs and problem types): the structure of the terminology; the construction of problems from integral solutions and integral coefficients (evidence that the problems have been borrowed rather directly from the mensuration tradition, without much further systematization or tinkering); and a tendency to construct solutions from sum and difference rather than semi-sum and semi-difference (as had been the Old Babylonian habit, and as Abū Bakr would mostly still do in the old problems).

42. For convenience I translate the propositions into symbols (it should be remembered that such a translation is always somewhat arbitrary—cf. the two different translations of prop. 7):

1. $\sqsubset\!\sqsupset(a,p+q+...+t) = \sqsubset\!\sqsupset(a,p) + \sqsubset\!\sqsupset(a,q) +...+ \sqsubset\!\sqsupset(a,t)$.
2. $\square(a) = \sqsubset\!\sqsupset(a,p) + \sqsubset\!\sqsupset(a,a-p)$.
3. $\sqsubset\!\sqsupset(a,a+p) = \square(a) + \sqsubset\!\sqsupset(a,p)$.
4. $\square(a+b) = \square(a) + \square(b) + 2\sqsubset\!\sqsupset(a,b)$.
5. $\sqsubset\!\sqsupset(a,b) + \square(^{a-b}/_2) = \square(^{a+b}/_2)$.
6. $\sqsubset\!\sqsupset(a,a+p) + \square(^{p}/_2) = \square(a+^{p}/_2)$.
7. $\square(a+p) + \square(a) = 2\sqsubset\!\sqsupset(a+p,a) + \square(p)$; or, alternatively, $\square(a) + \square(b) = 2\sqsubset\!\sqsupset(a,b) + \square(a-b)$.
8. $4\sqsubset\!\sqsupset(a,p) + \square(a-p) = \square(a+p)$.
9. $\square(a) + \square(b) = 2[\square(^{a+b}/_2) + \square(^{b-a}/_2)]$.
10. $\square(a) + \square(a+p) = 2[\square(^{p}/_2) + \square(a+^{p}/_2)]$.

We observe that prop. 6 coincides with prop. 5 if only $b = a + p$. Prop. 5 corresponds, however, to the situation where the sum of the two sides is known (as in prop. 9, a and b result from the splitting of a line in unequal segments), and where they are thus drawn in continuation of each other in the proof; prop. 6, on its part, is adapted to the situation where one exceeds the other by p, and the proof thus draws them in superposition. Precisely the same relation holds between prop. 9 and prop. 10, while prop. 4 and prop. 7 are similarly but not identically correlated.

43. Cf. note 26. It should perhaps be stressed once more that Savasorda's and Leonardo's use of propositions from *Elements* does not mean that they were employed within the tradition of mensuration algebra in the form we (and Leonardo and Savasorda) know them, only that they were still close enough to this tradition to be serviceable.

44. Strictly speaking, prop. 9 *is* cited, but in what seems to be an interpolated lemma. As pointed out by Ian Mueller, propositions 8 and 10 *might* have been cited in the same way, as justifications of unproved assumptions—*Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, p. 301 (Cambridge, Mass., & London: MIT Press, 1981). It seems as if the kind of knowledge contained in the three propositions was too familiar to require explicit citation once it had been proved.

45. They also solve problems about rectangles where the diagonal and either the sum of or the difference between the sides are known. As argued above (see note 38), at least one of these groups (most likely the two-square problems) will have belonged to the early phase of the mensuration algebra.

46. See my "*Dýnamis*" (cit. n. 5), where further references to work by earlier authors (not least Wilbur Knorr) on this question are given.

47. Since the second-degree problems which turn up in the first century (CE) Chinese *Nine Chapters on Arithmetic* (*Chiu chang suan shu. Neun Bücher arithmetischer Technik,* ed. trans. Kurt Vogel, pp. 91f (Braunschweig: Friedrich Vieweg & Sohn, 1968)) are related to the "new" Seleucid problems (and the dress of one of them, the leaning reed, an obviously borrowing), conquest can hardly be the only factor involved.

48. Part II, fol. 15^{r} (cit. n. 4).

49. Another suggestive deviation from Leonardo is Pacioli's version of Abū Bakr's N° 38 (above, p. 12): It is more correct than the Gherardo translation, which had been repeated so faithfully by Leonardo. Pacioli, indeed, finds the completing square 4 as "half the number of sides squared" (fol. 19^{r}). Since the Gherardo/Leonardo text is meaningless as it stands, it is highly unlikely that Pacioli could have used this version and just improved it. If he had done so (for example, supported by an *al-jabr* analysis), he could have produced a fully correct solution: instead, his explanation still presupposes tacitly that the excess and half the number of sides coincide.

We may infer that Pacioli's source for the pattern "sides and area" is thus not likely to have been the Gherardo version of the *Liber mensurationum*.

50. P. Nunez, *Libro de Algebra en Arithmetica y Geometria,* fol 277^{v}ff (Anvers: En casa de los herederos d'Arnaldo Birckman, 1567).