PREFACE

This volume in the Synthese Library Series is the result of a conference held at the University of Roskilde, Denmark, October 31st—November 1st, 1997. The aim was to provide a forum within which philosophers, mathematicians, logicians and historians of mathematics could exchange ideas pertaining to the historical and philosophical development of proof theory. Hence the conference was called Proof Theory: History and Philosophical Signi…cance. To quote from the conference abstract:

> Proof theory was developed as part of Hilberts Programme. According to Hilberts Programme one could provide mathematics with a …rm and secure foundation by formalizing all of mathematics and subsequently prove consistency of these formal systems by …nitistic means. Hence proof theory was developed as a formal tool through which this goal should be ful…lled.
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> It is well known that Hilbert's Programme in its original form was unfeasible mainly due to Gödel's incompleteness theorems. Additionally it proved impossible to formalize all of mathematics and impossible to even prove the consistency of relatively simple formalized fragments of mathematics by …nitistic methods. In spite of these problems, Gentzen showed that by extending Hilbert's proof theory it would be possible to prove the consistency of interesting formal systems, perhaps not by …nitistic methods but still by methods of minimal strength. This generalization of Hilbert's original programme has fueled modern proof theory which is a rich part of mathematical logic with many signi…cant implications for the philosophy of mathematics.
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> Although a completely secure justi…cation of mathematics is impossible it is, however, possible to achieve many fundamental partial results concerning relative consistency of theories, concerning strength of axiomatic systems and …nally concerning the relationship between constructive, predicative and classical systems of analysis.
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> The purpose of this meeting is to track the history of proof theory and its role in the analysis of the philosophical foundations of mathematics from its …rst primitive form in Hilbert's original Programme to its modern highly articulated form. Accordingly, the emphasis will be on historical and epistemological important episodes in the development of proof theory, not on technical aspects. All lectures will be of such a nature that they can be followed by mathematicians and philosophers without any professional training in proof theory but provided with general knowledge of fundamental issues.

The editors would like to thank the invited speakers including Prof. Solomon Feferman (Stanford University), Prof. Wilfried Sieg (Carnegie Mellon University), Prof. Dirk van Dalen (University of Utrecht), Prof. David Rowe (University of Mainz), Prof. Leo Corry (Tel Aviv University), Prof.

Moritz Epple (University of Mainz) and Prof. Erhard Scholz (University of Wuppertal) for contributing, in the most lucid and encouraging way, to the fulfillment of the conference aim. The editors are also grateful to the invited speakers for making their contributions available for publication.

The conference was organized by the Danish Network on the History and Philosophy of Mathematics. The editors would like to thank the network's organizing committee consisting of Prof. Kirsti Andersen (University of Aarhus), Prof. Jesper Lützen (University of Copenhagen), Dr. Tinne Hoff Kjeldsen (University of Roskilde) and the committee's secretary Lise Mariane Jeppesen (University of Roskilde). In turn, the network was only made possible by a research grant from the Danish Natural Science Research Council for which the editors and the network remain very grateful.

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INTRODUCTION

Proof theory is a technical sub-…eld of mathematical logic which originated during the intense discussions of the foundations of mathematics in the …rst half of this century. David Hilbert (1862-1943) and his co-workers created it, and the original main purpose of it was to secure once and for all the consistency of mathematics. This consistency proof should be based on such elementary methods that it would be transparent and evident to everybody from all sides of the mathematical "Grundlagenstreit". If this succeeded the foundational crisis of mathematics would come to an end and mathematicians could con…dently proceed with their creative activity.

This vision, which seems to be the message of Hilbert's famous paper "On the In…nite" from 1925, is not realizable. As is well-known the incompleteness theorems of Kurt Gödel (1906 -1978) from 1931 made it impossible to carry out Hilbert's program in its original form. That, however, did not lead to the death of proof theory. It already constituted an important mathematical discipline including a new method and approach to the philosophy of mathematics. It is, furthermore, important to notice that Hilbert was a very productive mathematician who contributed to nearly all areas of mathematics, i.e. geometry, algebra, number theory, mathematical physics, etc. His concerns were not only the consistency of formal systems. He had a much broader view of mathematics which his introduction of the method of ideal elements witnesses. The creation of proof theory must be seen as part of a broader discussion of the philosophy of mathematics and the development of new methods and theories around the beginning of this century.

It was the intention behind this conference to view the emergence of proof theory in a broad historical and philosophical context. On the one hand it should trace the development of proof theory and illuminate its growth into a mature theory. A mature theory which was deeply rooted in fundamental questions appearing during the development of new mathematical theories and methods around the turn of this century. New theories involving in…nite totalities (i.e. set theory and topology) and the introduction of the axiomatic approach to mathematics.

On the other hand it was it was the intention to discuss the philosophical signi…cance of proof theory for modern philosophy of mathematics. Some contemporary writers on philosophy of mathematics, the so-called "Maverick philosophers", seem to claim that proof theory, and other approaches within foundational studies, have been too in‡uential in philosophy of mathematics. By concentrating on narrow technical and logical issues mathematicians and philosophers have shifted the focus from viewing mathematics as the broad cultural activity it is to a cramped view of mathematics as a symbol manipulating activity only interested in its own internal coherence. The papers in this volume demonstrate in a brilliant way that such a criticism is a caricature. Not only does proof theory have a rich history as being part of the creation of modern mathematical logic, it also ‡ourishes both as a proli…c theory in itself and as an applied …eld in philosophy and in computer science. The conference had a series of three lectures by Solomon Feferman which, in a sense, introduces proof theory, highlights several of its most important results, and, …nally, presents some of the recent and most signi…cant results. So these lectures give the reader both a general introduction to proof theory and a very interesting and clear view of what is going on in contemporary proof theory.

Several papers dealt with the historical roots of Hilbert's proof theory and his thoughts on mathematics and its methods in general. These lectures constitute part two of this volume. Leo Corry discusses the roots of Hilbert's axiomatic approach to mathematics. It is interesting to see how this approach emerges out of Hilbert's early interests in geometry and mathematical physics. For instance, it is argued that Hilbert was in‡uenced by Heinrich Hertz (1857-1894), who in his book The Principles of Mechanics Presented in a New Form from 1894, described physical theories as "images". A good image should ful…ll three criteria, namely, permissibility, correctness and appropriateness. The origins of Hilbert's axiomatic systems, that is, requirements of completeness, consistency, independence, and simplicity, is de…nitely to be found in his acquaintance with the work of Hertz, Carl Neumann (1832-1925) and others. Furthermore, Corry demonstrates, as does Rowe, that consistency was not the main issue for Hilbert in the early years. The principal motivation for introducing the axiomatic method was to improve our understanding of mathematical and physical theories.

As mentioned above Hilbert's formalist program was only one part of his overall philosophy of mathematics. This becomes very clear from the paper by David Rowe. It is argued that foundations of geometry remained a more important …eld than logic and set theory at least up to about 1920. A central task Hilbert set himself was to explore the possibilities of constructing various geometries on foundations that did not require the axiom of Archimedes. He wanted to show how such geometries could be arithmetized rigorously. Consistency appeared as a kind of payo¤ of the arithmetization. Furthermore, it is also interesting to notice, as Rowe does, that Hilbert never reacted with any sense of alarm over the various paradoxes. Hilbert reported to Frege that Russell's discovery of the paradox in Frege's system was old news in Göttingen, that he himself had found even more convincing contradictions four or …ve years earlier. The real dedication to proof theory and simultaneously the beginning of the …nal phase in Hilbert's mathematical career started with Paul Bernays' arrival in Göttingen. After having realized that new techniques were necessary in order to prove consistency of arithmetics, Hilbert, Bernays and other mathematicians in Göttingen began to develop proof theory.

The technical development of proof theory is discussed by Wilfried Sieg. His contribution "Towards Finitist Proof Theory" provides an illuminating discussion of how the di¤erent mathematical and logical tendencies of the time motivated the formulation of Hilbert's program. Typically Hilbert's work has been divided into two periods, one ranging from 1900 - 1905 devoted to the problem syntactical consistency and yet another ranging from 1922 - 1931 devoted the …nitist foundations of mathematics. But Sieg points to the fact in between these two periods one can trace a third period from 1917 - 22 in which one …nds a sort of logicist reduction of mathematics. This is quite surprising given Hilbert's …nitist programmatic paper "Neubegrundung der Mathematik" from 1922. The motivation for this progression is analyzed in detail together with a discussion of how the diverse mathematical tendencies of the time were focussed into what we now know as Hilbert's program.

In the third part of this volume two other central …gures in the development of proof theory and the foundations of mathematics are discussed, namely L. E. J. Brouwer (1881-1966) and Hermann Weyl (1885 - 1955). Dirk van Dalen's paper discusses the development of Brouwer's intuitionism. van Dalen emphasizes the close connection between Brouwer's mysticism and his mathematical thoughts. In fact, Brouwer is a mysticist and a solipsist and his mathematical ideas are strongly related to his rather strange and vague mystic thoughts. Already in his dissertation from 1907 he derives the nature of mathematics from his mystical view of human nature. Dirk van Dalen shows how basic concepts of intuitionistic mathematics are related to and derived from Brouwer's mysticism. This is the case for concepts like free choice sequences and the creative subject. The introduction of free choice sequences "was the opening act of Brouwer's intuitionistic program" and in a series of papers starting in 1918 Brouwer introduces the fundamental concepts of intuitionism: spreads, species, the continuity principle, and the principle of bar induction. Based on these principles Brouwer develops, for example, a theory of the real continuum which di¤ers radically from the classical one. Given these principles it was possible for Brouwer, as van Dalen says, to turn intuitionistic mathematics into a strong system incompatible with classical mathematics. Later on Brouwer introduced additional principles in order to study the full continuum (based on free choice sequences), for instance, the concept of the creative subject which has been further studied in modern intuitionism (by Kreisel, Kripke, and others). Brouwer used this idea to strengthen weak counterexamples. Brouwer's active involvement in foundational studies ended in the late 20'ies. He withdrew "after the a¤ord of being thrown out of the editorial board of the Mathematische Annalen by Hilbert".

Brouwer based his philosophy of mathematics on a rather opaque mystical view of the world which makes it extremely di¢cult to understand his thoughts. But even if the mystical underpinning is removed and replaced with logically and epistemologically clearer ideas, as, for instance, done by S.C. Kleene, M. Dummett and others, the intuitionistic principles are still epistemologically problematic. That might be one major reason why Brouwer's intuitionism has not been more positively received by the mathematical community. This issue is taken up by Moritz Epple. He raises the question of whether Brouwer meets his own epistemological standards in his mathematical work. The conclusion of Epple's analysis is that Brouwer did not completely fulfill his own requirements. This is demonstrated through providing a very clear and thorough analysis of the principle of bar induction. This analysis is based on the formal examinations of intuitionism by Kleene and R. E. Vesley and on the philosophical analysis by M. Dummett. As Epple concludes, the reason why mathematicians did not accept intuitionism was not so much because the mathematical results were problematic, but because the epistemology which constituted a crucial part of his theory did not pay off. Nor did it convince the mathematical community of the philosophical superiority of Brouwer's intuitionism.

The other central figure who was discussed at the conference was Hermann Weyl. He is interesting for several reasons. Weyl was a mathematician with a deep knowledge of and interest in philosophy. And as Hilbert he contributed to many different fields of mathematics, for example, number theory, algebra, differential geometry, mathematical physics. His famous paper "Über die neue Grundlagenkrise der Mathematik" from 1921 in a sense started the intense "Grundlagensteit", or at least intensified it. He was deeply unsatisfied with the classical construction of the continuum and gave his own semi-constructive theory of it in the book Das Kontinuum from 1918. For a short time he was a follower of Brouwer's intuitionism but soon returned to a more classical view.

The paper by Feferman examines the mathematical content of Weyl's book Das Kontinuum. Weyl partially allied himself with Poincaré by not accepting the logicist program. He accepted the natural numbers as given and attempted a predicative construction of analysis on them. Feferman gives a modern formalization of Weyl's system. This formalization leads to the system ACA_0: In this system induction and comprehension are restricted to arithmetic properties. ACA_0 is a conservative extension of Peano Arithmetic. Within this system it is not possible to prove the Heine-Borel theorem which states that every bounded closed interval of reals is compact. Conversely it is possible to prove the Bolzano-Weierstrass theorem stating that every bounded sequence of reals contains a Cauchy sub-sequence. This of course restricts analysis. But elementary facts, as, for instance, the existence of Riemann integrals of stepwise continuous functions, can be established. Advanced analysis, to take an example, functional analysis, is beyond the reach of Weyl's system. However, as discussed in Feferman's last paper, a substantial part of modern analysis can be formulated in extensions of Weyl's system.

Hermann Weyl's dissatisfaction with the classical construction of the continuum and his characteristic ‡uctuating view on the foundations of mathematics are related and based on his broader view on philosophy and his concern with the foundations of di¤erential geometry. This is the topic of Erhard Scholtz's paper. It is important to notice that Weyl apparently has been in‡uenced by German idealistic philosophy, especially By Johann Gottlieb Fichte (1762-1814). Weyl was very unsatis…ed not only with the ordinary set theoretical construction of the continuum but also with his own semi-constructive continuum in Das Kontinuum. In both cases points were isolated atoms whereas, according to Weyl's intuition, points "are grown together in such a mode that it is not possible to single out isolated points, but only points endowed with a vaguely determined halo, a neighborhood. The intuitive continuum and the world of mathematical concepts are so distant from each other that the demand that both coincide has to be rejected as absurd". Scholtz demonstrates how Weyl's intuitive ideas of the continuum are rooted in Fichte's philosophy and in his re‡ection on concepts like space and matter in relativity theory. Apparently inspired by Fichte's Wissenschaftslehre Weyl tries to construct the outer world from in…nitesimal structures and matter from …elds. These ideas might be an essential reason why he turned his interests to Brouwer's intuitionism and to a ‡uent conception of the continuum. However, Weyl was receptive to criticism from physicists. During the late 20'ies his physical and ontological convictions changed and he admitted that Hilbert might have been right in his claim that formal trans…nite mathematics may become a useful tool in the "symbolic construction of the world". But he still considered the question of which kind of truth or objectivity that could be attributed to theoretical constructions that went far beyond any given reality. After the Second World War he again developed some serious doubts about the soundness of modern mathematical symbol construction.

Since the 1960'ies proof theory has developed into a forceful tool for the study of various approaches to the foundations of mathematics. In the last paper of this volume Feferman reviews various modern formal systems for constructive, predicative and classical analysis. As discussed by Epple, the intuitionism of Brouwer was looked upon with skepticism partly due to the fact that Brouwer's ideas and concepts were not su¢ ciently transparent and even epistemologically problematic. However, Errett Bishop initiated a form of constructive mathematics which was more acceptable for the general mathematician. It has been further developed by Douglas Bridges. Bishop and Bridges have managed to develop constructivism in a form which avoids con‡icts with classical mathematics as much as possible. This is carried out by only using "a¢ rmative" or "positive" concepts, by avoiding irrelevant de…nitions, i.e. arbitrary real functions, and by avoiding "pseudo-generality", i.e. never to work on non-separable spaces. In this way all theorems are as close to classical theorems as possible. For instance, Bishop introduced the

Limited Principle of Omniscience, LPO, about natural numbers

$$\forall n(f(n) = 0) \vee \exists n(f(n) \neq 0).$$

Then, if φ^* is a substitute of a classical theorem φ the following holds

$$LPO \wedge \varphi^* \rightarrow \varphi.$$

Feferman concentrates on Bishop's form of constructive mathematics. Feferman discusses several formal systems developed by himself which capture essential constructive and predicative principles. A leading idea behind introducing these systems is that they can be justified on basic grounds of one kind or another - i.e. constructive or predicative principles - and that mathematics can be developed in them. A consequence of this is that some of the systems go far beyond 2nd order systems. The first system Feferman introduces is W, called so in honor of Weyl. It is essentially a variable finite type extension of ACA_0. In W the LPO-principle holds and all the work of Bishop and Bridges can be directly represented in W. However, one cannot prove the principle of least upper bound in W. Feferman proposes the conjecture, that all (or almost all) scientifically applicable analysis can be carried out in W. It has been shown by Feferman and Jäger that W is conservative over the Peano Arithmetic and hence predicatively reducable. From this and Feferman's proposal it follows that the part of mathematics needed for science rests on completely arithmetical grounds. It is possible to show that even proof-theoretically weaker systems are sufficient for formalizing substantial parts of practical mathematics. That has been done within Reverse Mathematics as developed by Fiedman and Simpson. For instance, a substantial portion of functional analysis ban be carried out in the system WKL_0 which is obtained from ACA_0 by restricting the set existence axiom ACA to recursive predicates and adding König's lemma for binary branching trees.

The paper concludes with an interesting comment on the so-called Quine-Putnam indispensability argument which says that we must believe in the existence of those mathematical entities introduced in order to formulate our best scientific theories. Quine used this argument in order to justify Zermelo's set theory, Z, and to repudiate Zermelo-Fraenkel set theory, ZF. But Z is highly impredicative and much stronger than full 2nd order analysis. Hence, since most mathematics necessary for science can be developed in rather weak systems, Quine's acceptance of Z is based on an uncritical analysis of what is actually needed in mathematics for natural science. So the indispensability argument is useless unless it is followed by critical logical studies of what in fact is necessary and what can be reduced in those mathematical theories that are used in science.

Apparently it is not possible to find one epistemologically sound basis for modern mathematics that everybody would agree on. But modern proof theory has provided a huge variety of possible constructive and predicative alternatives to full classical analysis. In this way modern foundational research has led to a deeper understanding of such concepts as …nitism, constructivism, and predicativity and their role in organizing mathematical theories in epistemologically sound ways. As such this kind of research is essential to philosophy of mathematics.

The papers in this volume reveal in an impressive way the rich history of proof theory. They show how this theory emerged in a philosophically and scienti…cally extremely proli…c period and how it has contributed fundamentally to the shaping of modern philosophy of mathematics. Its historical signi…cance and actual importance for philosophy of mathematics have also been convincingly documented.

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