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INTRODUCTION

The study of epistemic attitudes – in particular knowledge and belief – dates at least back to the Scholaticism of the Middles Ages. The formal study of the same attitudes was then largely initiated by von Wright's seminal paper from the 1950's (37). The formal systematic study of knowledge and belief saw the light of day by Hintikka's book by the same name *Knowledge and Belief: An Introduction to the Logic of the Two Notions* from 1962 (16). Hardly a publication in epistemic logic has surfaced since without reference to this ground-breaking investigation. More recent monographs dedicated to epistemic and/or doxastic logic[1] include notably Lenzen (22), Schlesinger (25), Boh (3), Knuuttila (19), Meyer and van der Hoek (35), Fagin et al. (6), Sowa (30), Hendricks (12), (14), and Halpern (11).

The contemporary logics of knowledge and belief are advanced and sophisticated. Epistemic and doxastics logics for single agents have been catalogued; logics for multi-agent systems have been catalogued; epistemic modalities have been combined with temporal and alethic modalities, etc. These advances make the way for multiple epistemic operators, multiple doxastic operators, common knowledge operators, alethic and temporal operators, mono-modal systems, multi-modal systems, dynamic epistemic/doxastic systems, belief revision features and agents equipped with learning mechanisms. This is not an exhaustive list. There is a vast fan of important applications and models utilizing these powerful logics of knowledge and belief. Examples range from robots on assembly lines, social and coalitional interactions, card games, 'live' situations in economics, miscellaneous linguistic practices and so on.

It is not the purpose of this introduction to review epistemic logic from its date of birth to this day and age in detail.[2] Certain distinctive developmental features stand out as particularly pertinent to both the research progression and direction as well as the general epistemological and applicational relevance of epistemic logic. These may be subsumed under 'agent and system', 'active agenthood', 'multiple active agents', 'multi-modalites' and constitute the features with respect to which this introduction and the contributions in this volume are organized.

1. AGENT AND SYSTEM

The formal systematic surveys of epistemic and doxastic logics were early on largely influenced by the advances in (alethic) modal logic. Standard systems of modal logic were furnished with epistemic interpretations, and some fundamental results of epistemic logic could then be extracted.

Syntactically, the language of propositional epistemic logic is obtained by augmenting the language of propositional logic with a unary epistemic operator K_Ξ such that

$$K_\Xi A \quad \text{reads} \quad \text{`Agent } \Xi \text{ knows } A\text{'}$$

and similarly for belief

$$B_\Xi A \quad \text{reads} \quad \text{`Agent } \Xi \text{ believes } A\text{'}$$

for some arbitrary proposition A. These formalizations may be viewed as interpretations of $\Box A$ in alethic logic reading 'It is necessary that A'. Interpreting modal logic epistemically is crudely a reading of modal formulae as epistemic statements expressing attitudes like knowledge, belief or conviction of certain agents towards certain propositions.

The semantics of modal logic is likewise given a novel interpretation. Hintikka came up with a semantic interpretation of epistemic and doxastic operators respectively which in terms of standard possible world semantics may be rendered accordingly (16):

$K_\Xi A$: *in all possible worlds compatible with what* Ξ *knows it is the case that* A

$B_\Xi A$: *in all possible worlds compatible with what* Ξ *believes it is the case that* A

The basic assumption is that any ascription of propositional attitudes like knowledge and belief, requires partitioning the set of possible worlds into two compartments: The compartment consisting of possible worlds compatible with the attitude in question and the compartment of worlds incompatible with it. Based on the partition the agent is capable of constructing different 'world-models' using the epistemic modal language. The agent is not necessarily required to know which one of the world-models constructed is the real world-model. Be that as it may, the agent does not consider all these world-models equally possible or accessible from his current point of view. Some world-models may be incommensurable with his current information state or other background assumptions. These incompatible world-models are excluded from the compatibility partition.[3]

The set of worlds considered accessible by an agent depends on the actual world, or the agent's actual state of information. It is possible to capture this dependency by introducing a relation of accessibility, R, on the set of compatible possible worlds. To express the idea that for agent Ξ, the world w' is compatible with his information state, or accessible from the possible world w which Ξ is currently in, it is required that R holds between w and w'. This relation is written Rww' and reads 'world w' is accessible from w'. The world w' is said to be an *epistemic* or *doxastic alternative* to world w for agent Ξ depending on whether knowledge or belief is the considered attitude. Given the above semantical interpretation, if a proposition A is true in all worlds which agent Ξ considers possible then Ξ knows A and similarly for belief.

A possible world semantics for a propositional epistemic logic with a single agent then consists of a *frame* $\mathcal{F}$ which in turn is a pair (W, R) such that W is a non-empty set of possible worlds and R is a binary accessibility relation on W. A *model* $\mathbb{M}$ for an epistemic system consists of a frame and a denotation function φ assigning sets of worlds to atomic propositional formulae. Propositions are taken to be sets of possible worlds; namely the set of possible worlds in which they are true. Let *atom* be the set of atomic propositional formulae, then $\varphi : atom \longrightarrow P(W)$ where P denotes the powerset operation. The model $\mathbb{M} = \langle W, R, \varphi \rangle$ is called a Kripke-model and the resulting semantics Kripke-semantics (20): An atomic propositional formulae, **a**, is said to be true in a world w (in $\mathbb{M}$), written $\mathbb{M}, w \models \mathbf{a}$, iff w is in the set of possible worlds assigned to **a**, i. e. $\mathbb{M}, w \models \mathbf{a}$ iff $w \in \varphi(\mathbf{a})$ for all $\mathbf{a} \in atom$. The formula $K_\Xi A$ is true in a world w, i.e. $\mathbb{M}, w \models K_\Xi A$, iff $\forall w' \in W$: if Rww', then $\mathbb{M}, w' \models A$. The semantics for the Boolean connectives follows the usual recursive recipe. A modal formula is said to be *valid* in a frame iff the formula is true for all possible assignments in all worlds admitted by the frame.

Similar semantics may be formulated for the belief operator. Since a belief is not necessarily true but rather probable, possible, or likely to be true belief may for instance be modelled by assigning a sufficiently high degree of probability to the proposition in question and determining the doxastic alternatives accordingly. The truth-conditions for the doxastic operator are defined in a way similar to that of the knowledge operator and the model may also be expanded to accommodate the two operators simultanously.

A nice feature of possible world semantics is that many common epistemic axioms correspond to certain algebraic properties of the frame in the following sense: A modal axiom is valid in a frame if and only if the accessibility relation satisfies some algebraic condition. For an example, the axiom expressing the veridicality property that if a proposition is known by Ξ, then A is true,

$$K_\Xi A \to A, \tag{1}$$

is valid in all frames in which the accessibility relation is *reflexive* in the sense that

$$\forall w \in W : Rww.$$

Every possible world is accessible from itself. Similarly if the accessibility relation satisfies the condition that

$$\forall w, w', w'' \in W : Rww' \wedge Rw'w'' \to Rww''$$

then the axiom reflecting the idea that the agent knows that he knows A if he does,

$$K_\Xi A \to K_\Xi K_\Xi A, \tag{2}$$

is valid in all *transitive* frames. Other axioms of epistemic import require yet other rational properties to be met in order to be valid in all frames.

A nomenclature due to Lemmon (21) and later refined by Bull and Segerberg (4) is helpful while cataloguing the axioms typically considered interesting for epistemic logic (Table 1).

K	$K_\Xi(A \to A') \to (K_\Xi A \to K_\Xi A')$
D	$K_\Xi A \to \neg K_\Xi \neg A$
T	$K_\Xi A \to A$
4	$K_\Xi A \to K_\Xi K_\Xi A$
5	$\neg K_\Xi A \to K_\Xi \neg K_\Xi A$
.2	$\neg K_\Xi \neg K_\Xi A \to K_\Xi \neg K_\Xi \neg A.$
.3	$K_\Xi(K_\Xi A \to K_\Xi A') \vee K_\Xi(K_\Xi A' \to K_\Xi A)$
.4	$A \to (\neg K_\Xi \neg K_\Xi A \to K_\Xi A)$

TABLE 1. Common epistemic axioms

- Axiom K, also called the *axiom of deductive cogency*: If the agent Ξ knows $A \to A'$, then if Ξ knows A, Ξ also knows A'. The axiom maintains that knowledge is closed under material implication.
- Axiom D, also referred to as the *axiom of consistency* requires Ξ to have consistency in his knowledge: If an agent knows A, he does not simultaneously know its negation.[4]
- Axiom T, also called the *axiom of truth* or *axiom of veridicality*, says that if A is known by Ξ, then A is true.
- Axiom 4 is also known as the *axiom of self-awareness*, *positive introspection* or *KK-thesis*. They all refer to the idea that an agent has knowledge of his knowledge of A if he has knowledge of A.
- Axiom 5 is also known as the *axiom of wisdom*. It is the stronger thesis that an agent has knowledge of his own ignorance: If Ξ does not know A, he knows that he doesn't know A. The axiom is sometimes referred to as the *axiom of negative introspection*.
- Axiom .2 reveals that if Ξ does not know that he does not know A, then Ξ knows that he does not know not A.
- Axiom .3 maintains that either Ξ knows that his knowledge of A implies his knowledge of A' or he knows that his knowledge of A' implies his knowledge of A.
- Axiom .4 amounts to the claim that any true proposition *per se* constitutes knowledge and is sometimes referred to as *axiom of true (strong) belief*.

These axioms in proper combinations make up epistemic modal systems of varying strength depending on the modal formulae valid in the respective systems given the algebraic properties assumed for the accessibility relation.

The weakest system of epistemic interest is usually considered to be system **T**. The system includes T and K as valid axioms where K is valid in all Kripke-models. Additional modal strength may be obtained by extending **T** with other axioms drawn from the above pool altering the frame semantics to validate the additional axioms. By way of example, while $K_\Xi A \rightarrow A$ is valid in **T**, $K_\Xi A \rightarrow A$, $K_\Xi A \rightarrow K_\Xi K_\Xi A$ and $\neg K_\Xi A \rightarrow K_\Xi \neg K_\Xi A$ are all valid in **S5** but not in **T**. System **T** has a reflexive accessibility relation, **S5** has an equivalence relation of accessibility. The arrows in table 2 symbolize that the system to which the arrow is pointing is included in the system from which the arrow originates and hence reflect relative strength. Then **S5** is the strongest and **S4** the weakest of the ones listed.

Epistemic Systems			
KT4	=	**S4**	
KT4 + .2	=	**S4.2**	↑
KT4 + .3	=	**S4.3**	↑
KT4 + .4	=	**S4.4**	↑
KT5	=	**S5**	↑

TABLE 2. Relative strength of epistemic systems between **S4** and **S5**

One of the important tasks of epistemic logic have been, and still is, to map the possible complete systems of such logics hopefully allowing for a picking of the most 'appropriate' ones even though this appropriateness may be highly context-dependent as Halpern has noted (10). These 'appropriate' logics often range from **S4** over the intermediate systems **S4.2-S4.4** to **S5**. By way of example, Hintikka settled for **S4** (16), Kutschera argued for **S4.4** (36) , van der Hoek has proposed to strengthen knowledge according to system **S4.3** (34). In their contribution to this collection van Ditmarsch, van der Hoek and Kooi together with Fagin, Halpern, Moses and Vardi (6) assume knowledge to be **S5** valid.

In his contribution to this volume 'Knowledge, Belief, and Subjective Probability: Outlines of a Unified System of Epistemic/Doxastic Logic', Wolfgang Lenzen studies a range of axioms with respect to knowledge and two other attitudes expressing respectively conviction and belief, their formal properties and relations, philosophical implications and eventually concludes that:

> Summing up one may say that the logic of knowledge is isomorphic to an alethic modal system at least as strong as S4.2 and at most as strong as S4.4.

Conviction and belief are on the other hand weaker cognitive commitments than knowledge according to Lenzen. Even if the agent assigns probability 1 to the proposition in question and determines the set of alternatives accordingly this still does not suffice for validating the axiom of veridicality. The story is the same for belief as belief is considered weaker than conviction. For belief a proposition is considered likely or probable and the alternatives are considered doxastically accessible if the lower bound of subjective probability is greater than or equal .5 with respect to the proposition for the agent. Both conviction and belief imply a margin of error: The fact that Ξ is either convinced or believes A does not necessarily imply that A is true. Accordingly axiom T is dropped for conviction and belief and replaced by the consistency axiom D

$$B_\Xi A \rightarrow \neg B_\Xi \neg A \tag{3}$$

characterizing the consistency among convictions or beliefs. Lenzen then goes on to scrutinize additional axioms and systems appropriate for the two attitudes.[5]

One thing is to determine such approriate systems yet another is to determine appropriate or rational conditions of utterability for epistemic attitudes. Partially based on Grice's conversational maxims Lenzen formulates a number of pragmatic rules designed for the rational use of knowledge, belief and conviction governed statements. Lenzen provides a number of interesting philosophical insights in particular pertaining to the combined use of such commitments.

2. ACTIVE AGENTHOOD

A significant difference between alethic logic and epistemic logic is the addition of the agent Ξ to the syntax. One may legitimately question the roles assigned to the agents in epistemic logic. In the beginning they primarly served as indices on the accessibility relation between possible worlds. Epistemic-logical principles or axioms building up modal systems, however, are relative to an agent whom may or may not validate these principles. Being indices on accessibility relations will hardly suffice for epistemological and cognitive significance since there is nothing particularly epistemic about being indices.

Epistemic logics should accordingly observe the knower much more explicitly. An agent may have knowledge which is **S4.3** valid thereby obtaining a certain epistemic strength. An important set of questions seem to be *how* the agent has to *behave* in order to gain the epistemic strength that he has. To make epistemic logic pertinent to epistemology, computer science, artificial intelligence and cognitive psychology the agents must be activated. The original symbolic notation of a knowing agent also suggests this: An agent should be inside the scope of the knowledge operator—not outside as Hintikka notes in his contribution to this collection and elsewhere (17). Inquiring agents are agents who read data, change their minds, interact or have common knowledge, act according to strategies and play games, have memory and act upon it, follow various methodological rules, expand, contract or revise their knowledge bases, etc. all in the pursuit of knowledge. Inquiring agents are *active agents* (13).

The realization that the agents of epistemic logic should play an active role in the knowledge acquisition, validation, maintenance and potential interaction processes is discussed in Jaakko Hintikka's contribution to this volume 'A Second Generation Epistemic Logic and Its General Significance' as it in particular relates to his interrogative theory of inquiry developed here and elsewhere. Hintikka observes the obligation of active agenthood by emphasizing the strategies for his new application of epistemic logic as a logic of questions and answers and the search for the best questions to ask. Answers to questions are in essence requests for knowledge, information or epistemic imperatives in a strategic way inviting game-theoretical considerations:

> Another main requirement that can be addressed to the interrogative approach – and indeed to the theory of any goal-directed activity – is that it must do justice to the strategic aspects of inquiry. This requirement can be handled most naturally by doing what Plato already did to the Socratic *elenchus* and by construing knowledge-seeking by questioning as a game that pits the questioner against the answerer. Then the study of those strategies of knowledge acquisition becomes another application of the mathematical theory of games ...

Game theory is about strategies for winning games—and it is an agent whom may or may not have a winning strategy among other agents. van Benthem, Fagin, Halpern, Moses and Vardi, Aumann, Stalnaker and others studying game theory have demonstrated how logical epistemology uncovers important features of *agent rationality*. They also show how game theory adds to the general understanding of notions like knowledge, belief and belief revision.[6] Baltag, Moss, Solecki combine epistemic logic with belief revision theory to study actions and belief updates in games (2).

Mixing the theory of belief change and epistemic logic furnishes an illustrative example of active agents. The idea dates back to the mid 1990's. Alchourrón, Gärdenfors and Makinson's seminal belief revision theory (AGM) from the 1980's is a theory about the rational change of beliefs for expansions, contractions and revisions in light of new (possibly conflicting) evidence (1), (8). In 1994 de Rijke showed that the AGM-axioms governing expansion and revision may be translated into the object language of dynamic modal logic (5). Segerberg about the same time demonstrated how the entire theory of belief revision could be formulated in a modal logic.

A bit before but especially around the turn of the millennium Segerberg merged the static doxastic logic with the dynamics of belief change into 'dynamic doxastic logic' (27). Doxastic operators in the logic of belief like $B_\Xi A$ may be captured by AGM in the sense that 'A is in Ξ's belief-set T', or $\neg B_\Xi \neg A$ becomes '$\neg A$ is not in Ξ's belief-set T'. Similarly for other combinations of the belief operator with negation. An immediate difference between the two paradigms is that while AGM can express dynamic operations on belief-sets like expansions ('A is in Ξ's belief-set T expanded by D', i.e. $A \in T + D$), revisions ('A is in Ξ's belief-set T revised by D', i.e. $A \in T * D$), and contractions ('A is in Ξ's belief-set T contracted by D', i.e. $A \in T - D$), no such dynamics are immediately expressible in the standard language of doxastic logic. On the other hand, action languages include operators like $[\nu]$ and $\langle \nu \rangle$ which prefixed to a well-formed formula A, $[\nu]A$, respectively $\langle \nu \rangle A$ on Segerberg's interpretation mean that 'after $[every]$ $\langle some \rangle$ way of performing action ν it is the case that A'. By introducing three new operators $[+]$, $[*]$, and $[-]$ into the doxastic language the three dynamic operations on belief-sets may be rendered as $[+D]B_\Xi A$, $[*D]B_\Xi A$ and $[-D]B_\Xi A$.

After revising the original belief revision theory such that changes of beliefs happen in 'hypertheories' or concentric spheres enumerated according to entrenchment Segerberg has provided several axiomatizations of the dynamic doxastic logic together with soundness and completeness results (29), (28). The dynamic doxastic logic paradigm may also be extended to iterated belief revision[7] as studied by Lindström and Rabinowicz in (23) and accommodate various forms of agent introspection (24).

The AGM proposal for the rational change of beliefs has dominated the literature since its first formulation. In 'Economics and Economy in the Theory of Belief Revision' Hans Rott addresses the question of whether actual ecomomical principles and economic patterns have played any significant roles in the development of belief revision theory. Addressing this question requires a distinction between what Rott refers to as 'economic' behavior as opposed to 'economical' behavior. The former has close shaves with *economics* in terms of rationality and profit while the latter is more biased towards *economy* in the sense of thrifty or frugal. Tracing the history and motivation for the theory of belief revision Rott considers the extent to which these two different perspectives as they relate to cognitive changes and transformations have formed integral parts of the very formulation of AGM. His conclusion is striking and quite surprising:

> So belief revision theory has as a matter of fact not focussed on *economy*, and the idea of *economical* belief revisions has very limited normative force, too. Regarding economic belief revision, our findings are more encouraging. It is possible to reconstruct large parts of belief revision in terms of rational choice theory. As a matter of fact, ideas combining economics have prevailed in the AGM paradigm and related approaches.

3. MULTIPLE ACTIVE AGENTS

The logics of knowledge have been described with a single agent as the object so far. Active agenthood applies to multiple possibly interacting and coordinating agents and the formal framework considered may be expanded to a multi-agent setup.

Following Fagin, Halpern, Moses and Vardi (6) the idea is to syntactically augment the language of propositional logic with n knowledge operators, one for each agent involved in the group of agents under consideration. The primary difference between the semantics given for a mono-agent and a multi-agent semantics is roughly that n accessibility relations are introduced. A modal system for n agents is obtained by joining together n modal logics where for simplicity it may be assumed that the agents are homogenous in the sense that they may all be described by the same logical system. An epistemic logic for n agents consists of n copies of a certain modal logic. In such an extended epistemic logic it is possible to express that some agent in the group knows a certain fact, that an agent knows that another agent knows a fact etc. It is possible to develop the logic even further: Not only may an agent know that another agent knows a fact, but they may all know this fact simultaneously. From here it is possible to express that everyone knows that everyone knows that everyone knows, that ... . That is *common knowledge*.

A convention would hardly be looked upon as a convention if it was not for common knowledge among the agents to observe it as such Lewis once noted. Other norms, social and linguistic practices, agent interactions and games presuppose a concept of common knowledge. A relatively simple way of defining common knowledge is not to partition the group of agents into subsets with different common 'knowledges' but only to define common knowledge for the entire group of agents. Once multiple agents have been added to the syntax, the language is augmented with an additional operator C. CA is then interpreted as 'It is common knowledge among the agents that A'. Well-formed formulas follow the standard recursive recipe with a few, but obvious, modifications taking into account the multiple agents. An auxiliary operator E is also introduced such that EA means 'Everyone knows that A'. EA is defined as the conjunction $K_1A \wedge K_2A \wedge ... \wedge K_nA$.

To semantically interpret n knowledge operators, binary accessibility relations $R_{1,...,}R_n$ are defined over the set of possible worlds W. A special accessibility relation, R°, is introduced to interpret the operator of common knowledge. The relation must be flexible enough to express the relationship between individual and common knowledge. The idea is to let the accessibility relation for C be the transitive closure of the union of the accessibility relations corresponding to the singular knowledge operators. The model $\mathbb{M}$ for an epistemic system with n agents and common knowledge is accordingly a structure $\mathbb{M} = \langle W, R_1, R_2, ..., R_n, R^\circ, \varphi \rangle$ where W is a non-empty space of possible worlds, $R_1, R_2, ..., R_n, R^\circ$ are accessibility relations on W for which $R^\circ = (R_1 \cup R_2 \cup ... \cup R_n)^\circ$ where '$^\circ$' denotes the transitive closure and φ again is the denotation function assigning worlds to atomic propositional formula $\varphi : atom \longrightarrow P(W)$. The semantics for the Boolean connectives remain intact. The formula K_iA is true in a world w, i.e. $\mathbb{M}, w \models K_iA$, iff $\forall w' \in W$: if R_iww', then $\mathbb{M}, w' \models A$. The formula CA is true in a world w, i.e. $\mathbb{M}, w \models CA$, iff $R^\circ ww'$ implies $\mathbb{M}, w' \models A$. Varying the properties of the accessibility relations $R_1, R_2, ..., R_n$ as described above results in different epistemic logics. For instance system **K** with common knowledge is determined by all frames while system **S4** with common knowledge is determined by all reflexive and transitive frames. Similar results are possible to obtain for the remaining epistemic logics (6).

Common knowledge is not an unproblematic type of knowledge for interacting agents. In 'Common Knowledge Revisited', Ron Fagin, Joseph Halpern, Yoram Moses and Moshe Vardi consider a paradox arising in the context of common knowledge. On the one hand it seems that common knowledge among agents is a precondition for agreement and coordinated efforts of various kinds, and yet on the other such knowledge among agents in a certain group is actually impossible to obtain due to problems related to simultaneity and temporal imprecision:

> This puts us in a somewhat paradoxical situation, in that we claim both that common knowledge is a prerequisite for agreement and coordinated action and that it cannot be attained. We discuss two answers to this paradox: (1) modelling the world with coarser granularity, and (2) relaxing the requirements for coordination.

Active agenthood is also realizable directly on the agent level. One may also choose to endow the agents with *epistemic capacities* facilitating special epistemic behaviors. Fagin, Halpern, Moses and Vardi have for instance considered 'perfect recall' (6): Interacting agents' knowledge in the dynamic system may increase as time goes by but the agents may still store old information. The agent's current local state is an encoding of all events that have happened so far in the run. Perfect recall is sort of a methodological recommendation telling the agent to remember his earlier epistemic states.

There are other structural properties of agents being studied in the literature of dynamic epistemic logics. In an epistemic logic suited for modelling various games of imperfect information van Benthem refers to such properties as 'styles of playing' (31). Properties like 'bounded memory', various 'mechanisms for information updates' and 'uniform strategies' are analyzed in (32).[8]

In 'Concurrent Dynamic Epistemic Logic' Hans van Ditmarsch, Wiebe van der Hoek and Barteld Kooi study 'perfect recall' and 'no learning' as they relate to the change of knowledge over time given the execution of certain plans adding actions to the language of epistemic logic. For this new active and dynamic epistemic logic both soundness and completeness results are proved with the following broad perspective:

> Our work fits in approaches that not only dynamize the epistemics, but also epistemize the dynamics: the actions that (groups of) agents perform are epistemic actions. Different agents may have different information about which action is taking place, including higher-order information. This rather recent tradition treats all of knowledge, higher-order knowledge, and its dynamics on the same foot.

The epistemic logics of this day and age incorporate other modalities than the epistemic ones – in particular tense and temporal operators. Temporal imprecision is responsible for the common knowledge paradox as Fagin, Halpern, Moses and Vardi points out and have added temporal structure to the language of epistemic logic. van Ditmarsch, van der Hoek and Kooi make extensive use of tense as it relates to the transformation of knowledge, and adding actions to the language is yet another way of expanding the expressiveness of epistemic logic for both obtaining important epistemological and practical results using multi-modalities.

4. MULTI-MODALITIES

There are various ways in which multi-modalities may be realized in epistemic logic. One prevalent way is to follow Fagin et al. in reinpreting agents as well as possible worlds relative to entire system of agents (6).[9]

In a system of agents each individual agent is considered to being in some 'local state'. The whole system – as the sum of the local agents – is accordingly in some 'global state'. The dynamics may be modelled by defining what is referred to as a 'run' over the system which is a function from time to global states. The run may be construed as an account of the behavior of the system for possible executions. This gives rise to 'points' which are pairs of runs and times. For every time, the system is in some global state as a function of the particular time. The system may be thought of as series of runs rather than agents. What is being modelled are the possible behaviors of the system over a collection of executions. A system like this defines a Kripke-structure with an equivalence relation over points. The accessibility relation is specified with respect to possible points such that some point w' is accessible from the current point w if the agent is in the same local state at w and w'. Knowledge is defined with respect to the agents' local states. Truth of a formula is given with respect to a point. If truth is relative to a point then there is a question of *when* which opens up for the introduction of temporal operators. One may for instance define a universal future-tense operator such that a formula is true relative to the current point and all later points. The mixture of epistemic and temporal operators can handle claims about the temporal development of knowledge in the system.

In 'Laws, Facts and Contexts: Foundations of Multi-Modal Reasoning' John Sowa suggests an alternate route to embody multi-modalities. The idea is to drop possible worlds simpliciter and replace them with a concept of contexts elegantly represented by existential graphs derived from the works of notably Peirce and McCarthy. Possible world semantics are substituted for Dunn's semantics where a possible world is replaced by an ordered pair of laws and facts. Modality is now dependent on the choice of laws rather than on possible worlds and primitive accessibility relations between such often abstract entities. Based on these and additional prerequisites Sowa finally introduces 'nested graph models' which constitute a general formal framework with sufficient expressive power to handle multi-modalities and represent a variety of other semantics:

> [...] Dunn's semantics can use the axioms of an ontology as the laws that define the accessibility relations: Peirce-Kamp-McCarthy contexts combined with Tarski's metalevels can support metalevel reasoning about the selection of laws and facts; the outside-in evaluation method of Peirce's endoporeutic or Hintikka's game-theoretical semantics can accommodate the discourse constraints of dynamic semantics; and nested graph models are flexible enough to represent all of the above.

A different step in the direction of reinterpreting possible worlds for the sake of enhanced expressiveness is to follow Ryszard Wójcicki in his 'Referential Semantics' where possible worlds are understood as 'reference points'. Reference points may be construed as collections of agents using the same language and sharing or subscribing to the same criteria of truth. Such an interpretation has the advantage of dropping the metaphysical baggage that possible worlds tend to bring along and at the same time facillitates an alternative perspective on the concept of knowledge as a cross-referential phenomenon:

> Knowledge develops through critical assessment of alternative points of views, alternative often also in the sense that they are expressed in "different languages" meant to be languages based on different criteria of truth. ...

5. CONCLUSION

In the 1970's Scott noted that a mistake of modal logic was the focus on systems with only one modal operator (26). This narrow concentration is considered an obstacle for the extraction of pertinent results. For instance, if the epistemic operators are not combined with temporal operators little may be said about the change or development of knowledge and other epistemic states. The modern epistemic logic has come a long way since Scott's relevant concern. Active agenthood and multi-modalities have among other features added to the importance of (dynamic) epistemic logical modelling.

It is the hope that *Knowledge Contributors* will both shed light on the development of epistemic logic and demonstrate its relevance for applications in philosophy, computer science, game theory and other disciplines utilizing the means and methods of epistemic logic.

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NOTES

[1]To avoid tedious repetition the term 'epistemic logic' will occassionally be used to denote both the logic of knowledge and the logic of belief. Where the distinction is needed, the distinction is made.

[2]For excellent overviews of epistemic logic and its key issues refer to Gribomont and Gochet (9) and van Benthem (33).

[3]The epistemological significance of this partitioning is examined in (13) under the rubric of 'forcing'.

[4]This axiom also has a deontic motivation since if an agent is obligated to do whatever A prescribes, he is not at the same time obligated to do $\neg A$.

[5]Belief is often axiomatized between system KD4 and KD45.

[6]van Benthem has also pointed out that there is an epistemic logic 'hidden' in game-theory (31).

[7]A change in beliefs may either occur once in which case it is a one-shot revision or multiple changes may successively occur in which case it is an iterated revision.

[8]Agents as explcitly learning mechanims are also integral parts of Kelly's computational epistemology (18) and a related approach called modal operator epistemology (12), (14).

[9]A topological reconstruction of possible worlds furnishing the entities with enough formal structure to reason about them alethically, temporally and epistemically may be found in (12), (14).

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