

# BETWEEN THE FINITARY AND THE IDEAL

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Within contemporary philosophy of mathematics there is a trend focussing on how mathematics is done and how it evolves, rather than how it should be done or how it should evolve. This fact is somewhat contrary to the philosophy of mathematics in the 20th century, which to a large extent was dominated by views developed during the so-called foundational crisis in the beginning of that very century. These views have primarily focused on questions pertaining to the logical structure of mathematics and questions regarding the justification and consistency of mathematics. Mathematicians and philosophers like Gottlob Frege (1848–1925), Bertrand Russell (1872–1970), David Hilbert (1862–1943), Kurt Gödel (1906–1978) and others were very successful in their development of logic from around 1890 to, say, 1940 and they had a huge impact on the philosophy of mathematics of those days. Most probably it was Hilbert’s program, rise and fall, which was the most single influential factor of the foundational studies until 1960’s. Among the results was the widespread conception that the proper – if not the only – approach to philosophy of mathematics was through logic disciplines like set theory, proof theory, recursion theory and the like. This is no longer such a widespread opinion.<sup>1</sup> Today it is generally acknowledged that there is a fundamental incompleteness in mathematics – especially due to theorems of Skolem/Löwenheim and Gödel – which makes it impossible to characterise the mathematical activity. Logical aspects of mathematics are of course still important aspects for our understanding of what is going on, but they are truly not the only ones. Today it is also recognised that it is important to see mathematics as a cultural and historical phenomenon. Philosophical and historical perspectives in this direction show that there are aspects and parts of mathematics which are as ill founded and fallible as any other cultural scientific phenomenon. Certainly there are parts of mathematics in which truth seems to

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<sup>1</sup>Note however, that there still are views pointing in this direction. For instance Harvey Friedman argues that “As far as foundations of mathematics is concerned, the set theoretic interpretation of mathematical statements and mathematical proof is the currently accepted standard with no peer. [...] This set theoretical interpretation of mathematics is extremely coherent, and totally natural [...] In fact, set theory and its fragments appear to be the only fully rigorous totally natural completely coherent wide ranging powerful systematisation that we have throughout the whole intellectual landscape.” Friedman on FOM – an automated e-mail list for discussing foundations of mathematics – on August 29, 2002.

be of a very stable nature, but there are also instabilities and fluctuations to be found in mathematics. A satisfying philosophy of mathematics must be able to account for such matters.

In approaching such a philosophy I will argue in this paper that it is important to separate Hilbert's program from his general philosophy of mathematics. The latter is indeed prior to the program. But whereas the program is strongly damaged by the results of Gödel, Hilbert's general philosophy only needs to be slightly generalised in order to get a satisfactory philosophy of mathematics. It was a tendency to focus on Hilbert's program and relativisations hereof, rather than his philosophy, which led to the more traditional studies in the foundations of mathematics. On the other hand, I want to show that Hilbert's philosophy of mathematics is in harmony with the new trends. In doing this my intention is to show that there is a fruitful interplay between the logical analysis of mathematical structures and a more general philosophy of mathematics.

Finally, I will try to analyse what is to be understood under a generalised notion of 'finitary mathematics'.

## 1 Hilbert's Program and Conception of Mathematics

By the end of the 1880's Hilbert proved the (at the moment foremost) important result in the theory of invariance. The theorem is now known under the name Hilbert's Basis Theorem which has the following important consequence: If  $k$  is a field, then every ideal in the polynomial ring  $k[X_1, \dots, X_n]$  is finitely generated. The problem had been open for a long time, but Paul Gordon (1837–1912) – who was widely regarded as the leading authority working in the field – had, however, solved it for a special case and conjectured it to be true in the general case. Gordon's proof was algebraic and complicated—but constructive. Hilbert's general proof was short and elegant, but non-constructive, and it did represent a whole new and very powerful approach to the area. To be noted is also that Hilbert's techniques and results served as models for the subsequent developments in modern algebra. It is well-known that Gordon was more than just critical towards Hilbert's proof and methods in general, criticising them for being "theology" rather than mathematics. Truth it is that Hilbert's proof was highly non-constructive—he proved, that given an ideal it is not possible that a finitely generated basis does not exist. But it was not possible even to give bounds for the number of elements. Hilbert had to a certain extent a positive attitude towards the criticism, and tried later on to find more constructive versions of the proof. Nevertheless, he also continued to view the new methods of mathematics as very powerful and fruitful. Consequently, Hilbert already early in his career took part in the development of the modern methods of mathematics, and much of his success was based on very abstract notions and non-constructive proofs.

Now, Hilbert had a long mathematical career and was contributing, not

only in the theory of invariance, but nearly in all areas of mathematics such as algebra, geometry, number theory, mathematical physics, mathematical logic, etc. He had worked with the new mathematical theories and seen into the uncountable infinite that Cantor's set theory was about. He had seen and used the effectiveness and elegance of non-constructive methods. Thus, "Aus dem Paradies, daß Cantor uns geschaffen, soll uns niemand vertreiben können", as Hilbert (10, p. 170) puts it in his famous metaphor. Therefore he developed his program.

The program was based on his more general philosophy of mathematics, which was very Kantian at heart. Both the program and the philosophy were partly developed together with Paul Bernays (1888–1977) who began as the assistant of Hilbert around 1920. The philosophy was formulated in a very clear way in the lectures (14) Hilbert held in Göttingen 1919–1920, which were written up by Paul Bernays.<sup>2</sup> Their position was roughly the following. Mathematics can be split into two parts:

1. The finitary (or contentual) part of mathematics.
2. The ideal part of mathematics.

The rest of this paper will try to exploit what is to be understood under these terms.

## 2 Hilbert's Philosophy of Mathematics

The *finitary part of mathematics* was meant to be that part of mathematics of which there can be no doubts: Finitary reasoning about the natural numbers, i.e. no unrestricted quantifiers, and simple reasoning on finite graphs and geometrical figures. But what makes finitary reasoning special?

In December 1930 Hilbert gave a lecture in Hamburg. The lecture was given at a philosophical meeting and Hilbert starts by claiming that knowledge has three sources; logic and experience are the first and the second. The third, intuition, is described in the following way:

Wenn wir auch im einzelnen Kant heute nicht mehr zustimmen können, so behält doch der allgemeinste der Kantschen Erkenntnistheorie seine Bedeutung: jene anschauliche Einstellung a priori festzustellen und damit die Bedingung der Möglichkeit jeder Erkenntnis zu untersuchen. Ich meine, daß dies im wesentlichen in meinen Untersuchungen über die Prinzipien der Mathematik geschehen ist. Das Apriori ist dabei nichts mehr und nichts weniger als eine Grundeinstellung, die ich auch als die finite Einstellung bezeichnen möchte: es ist uns eben schon im voraus etwas in der Vorstellung gegeben: gewisse, außer-logische konkrete Objekte, die anschaulich als unmittelbares Erlebnis vor allem Denken da sind. Soll das logische Schließen sicher sein, so müssen sich diese Objekte vollkommen in

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<sup>2</sup>Edited with an English introduction by David E. Rowe in 1992.

allen Teilen überblicken lassen und ihre Aufweisung, ihre Unterscheidung, ihr Aufeinanderfolgen oder Nebeneinandergereihtsein ist mit den Objekten zugleich unmittelbar anschaulich gegeben als etwas, das sich nicht noch auf etwas anderes reduzieren läßt oder einer Reduktion bedarf. Dies ist die Grundeinstellung, die ich für die Mathematik wie überhaupt zu allem wissenschaftlichen Denken, Verstehen und Mitteilen für erforderlich halte, und ohne die eine geistige Betätigung gar nicht möglich ist. (12, p. 485–6)

It is clear, Hilbert inherits the fundamental concept of intuition from Kant. The intuition is the cognitive capacity which forms the sensations according to certain laws. Intuition [Anschauung] is a precondition for knowledge in general and science in particular and as such intuition has a determining (extra-logical) factor. The finitary mode of reasoning is some mode of reasoning which is fundamental for all mathematical thoughts – and as such it is secure and raised above all doubts. It is a foundation on which mathematics and science rest.

Tait has argued (24; 25) that finitism should be identified with what can be coded and justified by primitive recursive arithmetic. The class of functions which are primitive recursive is the smallest class:

1. Containing initial functions for zero, successor and projection.
2. Closed under composition. Also under primitive recursion which is: Given  $g, h$  we can define  $f$  as

$$\begin{aligned} f(x, 0) &= g(x), \\ f(x, y + 1) &= h(x, y, f(x, y)). \end{aligned}$$

Tait’s claim is twofold: A function is finitary if and only if it is primitive recursive. It is not difficult to argue for the “if”. In (16, p. 25-6) Hilbert and Bernays say “Unter einer [finite] *Funktion* verstehen wir hier eine anschauliche Anweisung, auf grund deren einer vorgelegten Ziffer, bzw. einem Paar, einem Tripel, ..., von Ziffern, wieder eine Ziffer zugeordnet wird”. A primitive recursive function is in this sense a finite function: It basically rests only on the two operations: substitution and iteration. Say that  $g$  is given and that we define  $f$  by  $f(0) = a_0$  and  $f(x + 1) = g(x, f(x))$ . Suppose we want to compute  $f(x)$ . If  $x$  is 0 then  $a_0$  is given. If  $x$  is not 0, then  $x = 1$  or  $x > 1$ . In the first case we have  $f(1) = g(1, a_0)$  which by assumption is finitarily computable—say the result is  $a_1$ . If  $x > 1$  then  $x = 2$  or  $x > 2$ . In the first case finitary reasoning gives us  $a_2$ . This process goes on until we reach  $x$ . The process is guaranteed to terminate as  $g$  is finitary and  $x$  is a (finitary) natural number.<sup>3</sup>

It is clear that a primitive recursive function is a finitary function. But are there more functions which are to be considered as finitary? What about Tait’s “only if”? In the programmatic article (10) Hilbert introduced the concept of a theory of functionals of finite type (together also with transfinite types).

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<sup>3</sup>Here we only have, by assumption, standard natural numbers.

Hilbert expected that these functionals could be used in connection with a proof of Cantor's continuum's hypothesis. In that connection Hilbert also mentioned the so-called Ackermann function. A modern theory building on Hilbert's type-theoretic ideas is system T of primitive recursive functionals of finite types. Let us introduce T and see how the Ackermann function is definable there.

## 2.1 Finite Type Theory

T is a type theory and thus it has a certain type structure. The ground type of T is type  $o$  which represents the natural numbers. Now, the types of T are generated inductively: if  $\sigma$  and  $\tau$  are types then  $\sigma \rightarrow \tau$  is also a type. Each type is thought of as representing a class of objects, and for  $o$  this is the natural numbers. Going to higher types,  $\sigma \rightarrow \tau$  is the type containing operations/functionals from  $\sigma$  to  $\tau$ . If  $F$  is a functional of type  $\sigma \rightarrow \tau$ , then we write this as is standard in mathematics  $F : \sigma \rightarrow \tau$ . Thus  $F : (o \rightarrow o) \rightarrow o$  is a functional taking an arithmetical function as an argument and gives a natural number. The question now is of course: Which objects inhabit T?

The language of T is multi-sorted, meaning that each symbol is assigned a certain type. It includes a symbol for the number zero,  $0 : o$  and a symbol for the successor function  $S : o \rightarrow o$ .<sup>4</sup> T has axioms concerning these objects stating for instance that  $0 : o$  is the first number and that  $S : o \rightarrow o$  is injective. T also has the standard axioms from propositional logic, but in order to do, among other things, primitive recursive arithmetic it has combinators  $k$ ,  $s$  and recursor  $R$ . The axioms concerning these are<sup>5</sup>

$$kxy = x \quad sxyz = xz(yz),$$

and

$$\begin{aligned} Rxy0 &= x \\ Rxy(Sz) &= y(Rxyz)z \end{aligned}$$

One can think of the equations as reduction rules: The terms on the left 'reduce' to the terms on the right, thus defining rules for calculations. As such it is seen that the prescribed operations are step-by-step calculable.

By using projector  $k$  and combinator  $s$  one can introduce the  $\lambda$ -operator as a *defined* notion. The operator behaves in the following way: if  $\lambda x.t[x]$  is a term of type  $\sigma \rightarrow \tau$  and  $s$  is a term of type  $\sigma$  then  $(\lambda x.t[x])s = t[s]$  where  $t[s]$  is of type  $\tau$ . With the lambda-notation it is simple to define basic arithmetical operations as for instance  $+$  which is of type  $o \rightarrow (o \rightarrow o)$ . A definition of  $+xy$ , or in more usual notation  $x + y$ , can be given by  $Rx(\lambda w, u.Swy)y$ .

In (10) a version of the Ackermann function is defined. Now we can come back to Tait's "only if". The Ackermann function is a computable function. It arises when one considers the following iterations. Let  $\varphi_1(a, b)$  be  $a + b$ . This

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<sup>4</sup>The intended meaning of  $S : o \rightarrow o$  is of course the mathematical function  $f : \mathbf{N} \rightarrow \mathbf{N}$  defined by  $n \mapsto n + 1$ .

<sup>5</sup>With respect to terms, parentheses are associated to the left; thus  $t_1 t_2 \dots t_n$  is short for  $(\dots((t_1 t_2) t_3) \dots t_n)$ .

surely is finitary. Let  $\varphi_2(a, b)$  be  $a \cdot b$  and let  $\varphi_3(a, b)$  be  $a^b$ . Furthermore,  $\varphi_4(a, b)$  is defined to be the  $b$ -th element of the sequence

$$a, \quad a^a, \quad a^{(a^a)}, \quad a^{(a^{(a^a)})}, \dots$$

The definition of  $\varphi_n$  is continued in this way. The Ackermann function is the number theoretic function  $\varphi$  which is defined as  $n \mapsto \varphi_n(n, n)$ . The function is not a primitive recursive function as it majorises any function which is primitive recursive, but intuitively it is computable. Is it finitary? It is definable in system T by recursion with function parameters. Let the iteration function be

$$I := \lambda f, x, y. \text{Ry}(\lambda u, v. f v) x.$$

Then the Ackermann function is definable by

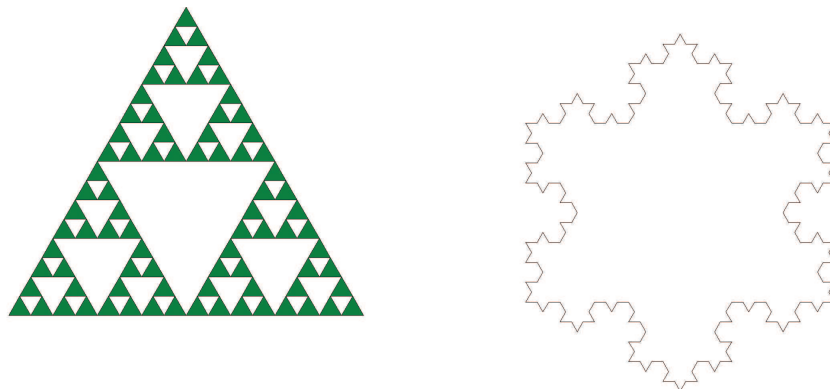
$$\varphi := \lambda x. \text{R}(Sx)(\lambda x, \psi, y. I \psi y y) x.$$

The interesting thing here is that, in fact Hilbert and his school clearly considered the function to be finitary. This has been shown by Zach (26). Moreover, we are inclined, due to an argument below, to consider the function as finitary, precisely due to the fact that it is definable in T. Therefore it seems that Tait's thesis – that finitism is identical with PRA – is to be refuted; at least with respect to Hilbert's conception of finitism. Below I will a further investigation of what is to be regarded as finitary.

## 2.2 Hilbert on Ideal Mathematics

*Ideal mathematics* was taken by Hilbert to be the highly abstract elements of mathematics of which the ontological status were not immediate. Examples of such ideal parts of mathematics could be (i) completed infinities, such as the set of natural numbers  $\omega$ , needed to develop a theory of the transfinite; (ii) the expansion of the real numbers by the complex number  $i = \sqrt{-1}$ , which enables us to prove the fundamental theorem of algebra; (iii) the 'completion' of Euclidean geometry by an infinite line consisting of points at infinity thus obtaining projective geometry, and so forth. As such, the ideal elements could not, according to Hilbert, be perceived by the senses. They had more the role of completing the finitary and regulating it—in the same way as in Kant's theory of knowledge where ideas of reason regulate knowledge. Note, that Kant indeed had two kinds of 'ideas': (i) The constitutive use of transcendental ideas causing paralogisms and antinomies of pure reason (19, A338-A567); and (ii) the regulative use of general ideas of pure reason, as used in science (19, A642-A704).

The ideal elements were supposed by Hilbert to be abstract elements introduced in the development of mathematics in order to simplify, generalise and complete already existing mathematics. But in such a process new mathematics would also arise and this was how Hilbert saw the expansion and progression of mathematics. Let us have a look at some further examples.



Fractals can be seen as ideal elements. Here we take the von Koch curve and the Serpinski triangle.

Both of the fractals are obtained by letting a generating algorithm ‘run’ to infinity. In the case of Serpinski we will denote the  $n$ -th approximation by  $S_n$ .  $S$  being the Serpinski triangle is the limit of this sequence of approximations. Now let  $C(S_n)$  be the circumference of  $S_n$  and  $A(S_n)$  be the area of  $S_n$ . It is clear that

$$A(S_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty$$

$$C(S_n) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty$$

In this way we see that no one-dimensional dimension seems to fit the Serpinski triangle. We are thus led to a generalised concept of dimension – the Hausdorff dimension  $\dim_H$  – in which

$$\dim_H(S) = \frac{\log 3}{\log 2} \simeq 1,585$$

and the von Koch curve:

$$\dim_H(K) = \frac{\log 4}{\log 3} \simeq 1,263$$

Apart from broadening our concepts, the fractals – being characterised by their broken dimensions – are useful and fruitful objects. Sometimes (or perhaps, in most cases) they give rise to a deepening of more ‘basic’ mathematical concepts. Thus the von Koch curve was introduced in order to scrutinise our concept of continuity. From the curve  $K$  it was realised that it was possible for a continuous curve to be nowhere differentiable. This came as a big surprise in the late 19th century.

Another example of ideal elements in mathematics are generating functions.

The Fibonacci numbers, probably introduced by Leonardo de Pisa around 1200, will serve as an example. Here a purely number theoretic problem is easily solved when translated into complex analysis. The sequence of the Fibonacci numbers is recursively defined as

$$f(0) = 0, f(1) = 1 \text{ and for } n \geq 2, f(n) = f(n-1) + f(n-2).$$

If we let  $\chi$  be the characteristic function of the predicate “being equal to 1”, i.e.,

$$\chi(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases}$$

and let  $\psi$  be the characteristic function of the predicate “being strictly greater than 0” then we can define the (slightly generalised) Fibonacci numbers in one formula:

$$\forall n \in \mathbf{Z}: f(n) = \psi(n)(f(n-1) + f(n-2) + \chi(n)) \quad (1)$$

The problem which now interests us is how to find a direct expression for the  $n$ -th Fibonacci number. In order to compute the  $n$ -th number we will, given the definition, have to compute the whole sequence of numbers between 0 and  $n$  in order to compute  $f(n)$ . This is of course very inefficient and we would thus like to have a closed expression telling us how to compute directly. This is a purely number theoretic question but the trick which makes the solution particularly easy and elegant is to embed the problem in complex analysis. Specifically we will look at the so-called generating function for the Fibonacci sequence.

In order to introduce and use generating functions we need some complex analysis. For any sequence  $(a_k)_{k \geq 0}$  of complex numbers we can form the *formal power series in  $z$  with coefficients  $a_k$* . This is defined as

$$P(z) := \sum_{k=0}^{\infty} a_k z^k := \left( \sum_{k=0}^n a_k z^k \right)_{n \geq 0},$$

$z \in \mathbf{C}$ . Thus  $P(z)$ , which is also called the generating function for  $(a_k)$ , is a sequence of finite sums. If the sequence  $(\sum_{k=0}^{\infty} a_k z^k)$  converges then the formal power series converges. Whether or not formal power series converges we have certain operations we can perform on them. If  $P(z)$  and  $Q(z)$  are formal power series with coefficients  $(a_k)_{k \geq 0}$  and  $(b_k)_{k \geq 0}$ , respectively then we can add  $Q$  to  $P$  and obtain  $(P+Q)(z)$  with coefficients  $(a_k + b_k)_{k \geq 0}$ . Likewise we can multiply with a scalar, form a certain Cauchy product and so on.

When we return to the Fibonacci numbers the formal power series with these numbers as coefficients is:

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n, \quad z \in \mathbf{C}.$$

When the recursive definition of the Fibonacci numbers (1) is used we get

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} (\psi(n)(f(n-1) + f(n-2) + \chi(n)))z^n \\
&= \sum_{n=0}^{\infty} f(n-1)z^n + \sum_{n=0}^{\infty} f(n-2)z^n + \sum_{n=0}^{\infty} \chi(n)z^n \\
&= \sum_{n=0}^{\infty} f(n)z^{n+1} + \sum_{n=0}^{\infty} f(n)z^{n+2} + z \\
&= z \sum_{n=0}^{\infty} f(n)z^n + z^2 \sum_{n=0}^{\infty} f(n)z^n + z \\
&= F(z)(z + z^2) + z.
\end{aligned}$$

If we isolate  $F(z)$  in this equation we find that

$$F(z) = \frac{z}{1 - z - z^2}$$

This is a well-defined function whenever  $1 - z - z^2$  is different from 0. If  $|z|$  is close to 0 this function is analytical and can therefore be developed as a power series using for instance Taylor polynomials. There is, however, something which is more elegant in our case. For  $\gamma = \frac{1+\sqrt{5}}{2}$  this can (using fairly simple methods) be re-written as

$$F(z) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \gamma z} - \frac{1}{1 + \frac{z}{\gamma}} \right).$$

The body of the parenthesis can be developed as series:

$$\frac{1}{1 - \gamma z} = \sum_{k=0}^{\infty} (\gamma z)^k, \quad \frac{1}{1 + \frac{z}{\gamma}} = \sum_{k=0}^{\infty} \left( \frac{-z}{\gamma} \right)^k$$

Putting things together we get

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}} (\gamma^k + (-1)^{k+1} \gamma^{-k}) z^k.$$

Therefore, by leaving the complex terms aside we get a closed expression for the  $n$ -th Fibonacci number

$$f(n) = \frac{1}{\sqrt{5}} (\gamma^n + (-1)^{n+1} \gamma^{-n}).$$

Interestingly, this also makes us able to say something about the asymptotic behaviour of  $f(n)$ , namely when  $n$  is large then  $f(n)$  is approximately  $\frac{1}{\sqrt{5}} \gamma^n$ .

This example is nothing but a special case of a much more general well-known theorem:

**Theorem.** Suppose  $d \in \mathbf{N}$ , and let  $q_1, \dots, q_d \in \mathbf{C}, q_d \neq 0$ . Put  $q_0 := 1$  and let  $\alpha_1, \dots, \alpha_e$  be the roots of the polynomial  $q(z) = \sum_{k=0}^d q_k z^{d-k}$ . Then for any sequence of numbers  $(a_k)_{k \in \mathbf{N}}$  in the following are equivalent:

1. The sequence is recursively given, that is for every  $k \in \mathbf{N}, k > d$

$$a_k + q_1 a_{k-1} + q_2 a_{k-2} + \dots + q_d a_{k-d} = 0$$

2. There are polynomials  $g_1(z), \dots, g_e(z)$  where  $\text{grad}(g_i) < c$ , for a certain  $c \in \mathbf{N}, i = 1, \dots, e$  such that for all  $k \in \mathbf{N}$

$$a_k = \sum_{i=1}^e p_i(k) \alpha_i^k$$

The proof of theorem reveals a method for finding the explicit expression and this method uses generating functions as in the example of the Fibonacci numbers. Here we use generating functions as a very powerful tool for solving questions of a much simpler character. But it is the very translation of a problem in a simple domain in to a complex domain which makes the solution particular easy.

### 2.3 Hilbert's Justification

As seen by Hilbert the ideal elements of the mathematical method and universe were, of course, of indispensable value. However, Hilbert was at the same time aware of the fact that this progressiveness of the mathematical method was transcending the secured finitary parts of mathematics and it was therefore in need of some kind of justification.

This justification would consist in showing, mathematically, that the ideal part of mathematics could not prove new purely finitary statements, i.e. could not prove finitary statements which were not provable already in the finitary part of mathematics. This is where Hilbert connected the axiomatic approach, the idea of a 'proof theory', with his general view on mathematics as just described. In modern technical terms the goal of the program is described by the following.

Let  $S$  be some formal system representing mathematics—both the ideal and the finitary part. A formula in the language of  $S$  is a finite object and it can therefore be coded effectively by a natural number; proofs in  $S$  can likewise be coded. Thus,  $\text{Proof}_S(x, y)$  is a predicate obtaining between two natural numbers  $x$  and  $y$  expressing that  $x$  encodes a proof in  $S$  of some formula having code  $y$ . As is standard, let  $\ulcorner A \urcorner$  be the code of  $A$ . (The argument below is in the given form a little vague—the details rely on the specific properties of the encoding; see (23, sect. 2–4).) In technical terms the essence of Hilbert's program was that for any finitary statement  $R(x)$  with  $x$  as free variable the reflection principle

$$\text{Proof}_S(u, \ulcorner R(\dot{x}) \urcorner) \rightarrow R(x) \tag{2}$$

should be provable by finitary means (where  $\dot{x}$  refers to the  $x$ -th numeral). However, it would be sufficient to establish consistency of S in a finitary way. For suppose one has a proof in S of some finitary statement  $R(x)$  containing only  $x$  as free variable, hence

$$\text{Proof}_S(u, \ulcorner R(\dot{x}) \urcorner) \tag{3}$$

would be finitarily provable. However, if  $R(x)$  were not true for all  $x$  then for some  $c$ ,  $\neg R(c)$  would be provable within S. In fact we would have, due to  $\Sigma_1$  completeness,

$$\neg R(x) \rightarrow \text{Proof}_S(v_x, \ulcorner \neg R(\dot{x}) \urcorner), \tag{4}$$

where  $v_x$  depends on which value  $x$  takes. If, on the other hand, we could prove consistency of S by finitary means we would have

$$\neg( \text{Proof}_S(u, \ulcorner R(\dot{x}) \urcorner) \wedge \text{Proof}_S(v, \ulcorner \neg R(\dot{x}) \urcorner) ) \tag{5}$$

Now, (2), (4) together with 5 implies that  $\neg \neg R(x)$  has a finitary proof, and since  $R(x)$  is a finitary statement this implies  $R(x)$ .

The argument is modern version of Hilbert's argument as found, for instance, in (11, p. 78):

Aber auch wer sich mit der Widerspruchsfreiheit nich begnügt und noch weitergehende Gewissenskrupel hat, muß die Bedeutung des Beweises der Widerspruchsfreiheit anerkennen, nämlich als einer allgemeinen Methode aus Beweisen für allgemeine Sätze vom Charakter etwa des Fermatschen Satzes, die mit Hilfe der  $\epsilon$ -Funktion geführt sind, finite Beweise zu gewinnen.<sup>6</sup>

On page 78–79 Hilbert then goes on to explain – by use of Fermat's last theorem as an example – how this can be accomplished. In that argument Hilbert also uses  $\Sigma_1$  completeness, although he due to his time does not recognise it as something special. Note, this also throws light on the late Hilbert's conception of the connection between consistency and existence.

Consequently, Hilbert focused on the program of proving consistency of mathematics in order to provide a finitary foundation for ideal mathematics and thereby justify the use of ideal elements. Bernays sums up the advantage and goal of such a program:<sup>7</sup>

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<sup>6</sup>The  $\epsilon$  operator was a technical invention within proof theory, and Hilbert saw it as representing a highly ideal aspect of mathematics.

<sup>7</sup>Bernays uses the words *anschauliche Erkenntnisse* which belong to the Kantian theory of knowledge. These are often translated into English by *intuitive knowledge*. *Intuitive* here refers to *intuition* which is the translation of *Anschauung*. Thus, *intuitive knowledge* is knowledge based on intuition (*Anschauung*) and, consequently, it does not necessarily mean *immediate*. It rather means that when such knowledge is obtained it is by the way we obtain it objective. The justification of the ideal elements that Bernays refers to would thus be absolute.

Unter diesem Gesichtspunkt werden wir versuchen, ob es nicht möglich ist, jene transzendenten Annahmen in einer solchen Weise zu begründen, daß nur *primitive anschauliche Erkenntnisse zur Anwendung kommen*. (2, p. 11).

Moreover, Bernays stresses the importance that the central problem becomes a *mathematical* problem:

Gerade darin liegt der große Vorzug des Hilbertschen Verfahrens, daß die Probleme und Schwierigkeiten, welche sich in der Grundlegung der Mathematik bieten, aus dem Bereich des Erkenntnistheoretisch-philosophischen in das Gebiet des eigentlich Mathematischen übergeführt werden. (2, p. 19).

It is important to note that underlying the program is the idea of separating the foundational studies of mathematics from philosophy and epistemology. Of course, the characterisation of finitism is motivated by a philosophical analysis—but when that is done *it is up to the mathematicians to take care of their own foundation*, “gerade wie ja auch der Astronom die Bewegung seines Standortes berücksichtigen, der Physiker sich um die Theorie seines Apparates kümmern muß und der Philosoph die Vernunft selbst kritisiert” as Hilbert (9, p. 155) expresses it.

### 3 Failure of Hilbert’s Program

Gödel, though not a student of Hilbert, was attracted to Hilbert’s foundational questions and problems. Around 1930 he proved, within less than a year, not only the completeness of first order predicate logic,<sup>8</sup> but he also showed – when pursuing the program – that Hilbert’s original program was unattainable. Gödel (5) showed that for any consistent theory  $T$  containing just a minimum of arithmetic and given by a recursively enumerable set of axioms – and certainly this was included in Hilbert’s finitism – two things are the case:<sup>9</sup>

**Theorem 1.**  $T$  is syntactically incomplete in the sense that there exists a sentence  $A$  in the language of  $T$  such that  $T$  does not prove  $A$  nor  $\neg A$ .

**Theorem 2.** One cannot in  $T$  prove the consistency of  $T$ .

The theorems are a disaster for Hilbert’s program. First of all, they show that there is no absolute proof of the consistency of all mathematics: The reduction of ideal elements used in proofs of finitary statements cannot be obtained by a finitary consistency proof, since such a proof does not exist.

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<sup>8</sup>A problem formulated by Hilbert and Ackermann (15).

<sup>9</sup>Gödel actually assumed the slightly stronger  $\omega$ -consistency, but J.B. Rosser later replaced this by consistency. Furthermore, theorem 2 was stated by Gödel (5) without proof. The first published proof of the theorem is in (17).

Secondly, it was more than an implicit assumption of Hilbert and his co-workers that the formal systems were representing mathematics completely. As late as 1930 they conjectured that Peano arithmetic was deductively closed, in the sense that adjoining any sentence not provable in Peano arithmetic would make it inconsistent, see (3, 59). However, this conjecture is certainly refuted by theorem 1, and this has severe consequences for the program. Due to theorem 1 any consistent formalisation  $S$  of any mathematical theory will be incomplete in the sense, that there will be a sentence  $A$  true in the standard model but  $A$  is undecided by  $S$ .<sup>10</sup> Moreover,  $S + A$  will also be incomplete, and such an iterated process of adding undecidable formulas will go on forever. Therefore, there is no obvious way to choose an  $S$  and such a choice becomes a *central* question.

All this shows that the *global* reduction – to use a phrase of Solomon Feferman (4) – Hilbert had in mind is impossible. But maybe a more *local* reduction is possible?

However, after theorem 2 there is no obvious choice of a constructive part of mathematics to which some of ideal mathematics can be based on or perhaps reduced to. But maybe Hilbert’s finitism was only a first approximation of constructivism. In any case we see that the separation of foundational studies from philosophy and epistemology, which Hilbert aimed at, is lost. Questions regarding the nature of mathematical concepts and how we come to have knowledge about them are certainly not eliminated as Hilbert hoped. And what constructivism is, or how a satisfactory interpretation of constructivism can be strong enough to interpret – in some way or another – parts of ideal mathematics are very open ended questions after Gödel (5).

Let me try to give two examples of what I consider as evidence for some finitary parts of mathematics.

## 4 What is Finitary Mathematics?

Coming back to our system  $T$  let me give some reasons why functionals of  $T$  should be considered as finitary. For sure, one can define fairly complicated functionals in  $T$ . But the more complicated functionals are always defined *inductively* by a chain of definitions where each step defines a new functional in terms of previously defined ones. Now, the single steps describe simple calculations. However, given a concrete well-formed closed term we cannot directly read of how many calculations we have to perform before the overall calculation is done. But, by (non-constructive) mathematical reasoning we can, of course, give bounds for any such term, but this is in our case perhaps not that interesting

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<sup>10</sup>Such sentences need not to be artificial and without mathematical meaning. Paris and Harrington (22) gave an example of a finite version Ramsey’s theorem which is independent of Peano arithmetic. Later Kirby and Paris (20) gave a simpler independence result concerning so-called Goodstein sequences. Recently H. Friedman has in a series of papers (which can be downloaded at <http://www.math.ohiostate.edu/~friedman>) provided theorems from finite graph theory, which are even unprovable in predicative mathematics, e.g. the Graph Minor Theorem.

since it is the same (non-constructive) mathematics we want to justify. However, we are given inductive rules in order to do the computation—intuitively we just do not know how many computations we will have to perform.

Therefore, when we define new procedures (or functionals) out of previously defined ones by the schemes we get a calculable functional, since the equations defining  $k$ ,  $s$  and  $R$  prescribe constructive operations. Epistemologically, we consider it as a general human ability to carry out such inductive operations and consequently, we find it justified to consider Gödel's functionals as constructive.

## 4.1 Reductions

Given we have accepted  $T$  as constructively unproblematic the next task is to consider ideal elements which can be given an interpretation relative to  $T$ . We will analyse the following principles:

1. *Extensionality*.
2. *Markov's principle*: If  $A(x)$  is decidable for any  $x$  and if  $\neg\neg\exists xA(x)$  holds then  $\exists xA(x)$ , i.e.

$$\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists xA(x) \rightarrow \exists xA(x).$$

3. *Independence-of-premise* (for certain classes of formulas). If  $A \rightarrow \exists yB(y)$  holds and  $y$  is not a free variable of  $A$  then  $\exists y(A \rightarrow B(y))$  holds, i.e.

$$(A \rightarrow \exists yB(y)) \rightarrow \exists y(A \rightarrow B(y)).$$

To see why, say, 3 is constructively doubtful let the following be noted. The constructive reading (which is the interpretation given by Kolmogorov and Heyting) of  $A \rightarrow \exists yB(y)$  is that given a proof of  $A$  we can construct an object  $t$  and proof of the fact that  $t$  has property  $B$ , i.e.  $B(t)$ . The principle says that in case  $A \rightarrow \exists yB(y)$  holds then in fact  $\exists y(A \rightarrow B(y))$ . But the constructive reading of the conclusion is that we can construct an object  $t$  such that given any proof of  $A$  we can prove  $t$  to have property  $B$ . Thus the principle says that  $t$  can be constructed independently of the proof of  $A$ , and this is not constructively evident given  $A \rightarrow \exists yB(y)$ .

One of our main purposes here is to sketch an analysis of 1–3 in order to provide coherent and consistent systems that partly contain these *prima facie* non-constructive/non-finitary principles. Systems that nevertheless can be seen as constructively meaningful. A system is constructively meaningful when it has several important properties, say the existence property, in a crucial way. In the case at hand this means, for instance, that when an existential statement is proved in one of the theories then there is a number or a function from Gödel's system  $T$  actually realising the existential quantifier. And this realiser can be found given any proof of the statement. Thus existence in the disputed theories is shown to be based on objects coming from  $T$  which we take to be constructive.

The general idea is that given the conceptual acceptance of a certain system, what can (or maybe should) we then accept on the basis of this; in other words, Given the acceptance of some system  $S_0$  (which in our case is  $T$ ), if  $S_1$  reduces to (is interpretable in)  $S_0$ , then we also accept  $S_1$ .

The concept of *reduction* or *interpretation* that we work with here is the following:

$S_1$  reduces to  $S_0$ , in symbols  $S_1 \preceq S_0$  if

- (i)  $S_1$  has the existence property and the disjunction property.<sup>11</sup>
- (ii) The terms realising (i) live in  $S_0$ .

If we accept  $S_0$  and have  $S_1 \preceq S_0$  then we consequently accept  $S_1$  on the basis of  $S_0$ , since  $S_1$  has various nice properties and the realisers which make this possible come from  $S_0$ , found to be constructively unproblematic.

## 4.2 Specific Reductions

We will now take a look at some specific ideal elements. First extensionality. In any of the following formal theories we have equality only between objects of the lowest type—that is, equality between numbers. Functions and functionals are equal if they are equal on equal arguments. Equality in higher types is thus *defined* in terms of lower types and is therefore extensional. As the types can be fairly complicated it is not obvious that our functionals respect this defined notion of equality. However, we can introduce axioms which claim all functionals to respect it. Let  $\sigma$  be type  $\sigma_1 \rightarrow (\dots \rightarrow (\sigma_n \rightarrow o)\dots)$ . Then  $z$  of type  $\sigma$  is extensional, i.e. respects equality iff

$$\forall x_1^{\sigma_1}, y_1^{\sigma_1}, \dots, x_n^{\sigma_n}, y_n^{\sigma_n} \left( \bigwedge_{i=1}^n x_i = y_i \rightarrow z\mathbf{x} = z\mathbf{y} \right)$$

where  $\mathbf{x}$  denotes  $x_1, \dots, x_n$ . Claiming full extensionality is to claim that all functionals of the theory are extensional. However, it turns out that we cannot always interpret full extensionality. In certain situations we can interpret only *weak* extensionality. This is not introduced by axioms but rather as a rule which within natural deduction has the following form:

$$\frac{\begin{array}{c} \Delta \\ \vdots \\ s = t \end{array}}{r[s] = r[t]}$$

where  $\Delta$  consists only of quantifier free formulas, and  $s, t$  and  $r[x]$  are terms.<sup>12</sup>

<sup>11</sup>A theory  $S$  is said to have the existence property in case the following rule holds:  $S \vdash \exists x A(x) \Rightarrow S \vdash A(t)$  for some term  $t$ . Likewise,  $S$  has the disjunction property if we have that for any provable closed formula  $A \vee B$  there is a term deciding whether  $A$  or  $B$  holds.

<sup>12</sup>Full extensionality could also be formulated as a rule instead of using axioms. Then it has the same form as weak extensionality, just without any restrictions on the assumptions.

In the following there will be two different theories on top of which we will put ideal principles. They are two different versions of intuitionistic number theory generalised to finite types; one will have full extensionality,  $E\text{-HA}^\omega$ , and the other one will only have weak extensionality,  $WE\text{-HA}^\omega$ . Different elements which are generally non-constructive will then be added to these theories. First of all, two different versions of the principle of independence-of-premise. The principle in its general form was defined on page 14. Here we will work with the case where the formula  $A$  does not contain any existential quantifier, nor any disjunction—this will be denoted  $IP$ . As it turns out, this principle is constructively in conflict with Markov’s principle. Thus, we will have to consider a more restricted version, namely the case where the formula  $A$  is purely universal. This principle will be denoted  $IP_\forall$ . Markov’s principle is found on page 14. However, in our case the formulas which are decidable are the quantifier free formulas. Consequently, the form of Markov’s principle ( $MP$ ) which we will look at is

$$\neg\neg\exists x A_{\text{qf}}(x) \rightarrow \exists x A_{\text{qf}}(x),$$

where  $A_{\text{qf}}$  is quantifier free.

Finally, the axiom of choice, denoted  $AC$ , is also added.

Let  $\Gamma$  be any arbitrary but fixed set of true existence and disjunction free sentences and likewise, let  $\Gamma_\forall$  be any arbitrary but fixed set of true purely universal sentences. Then we define the following theories:

$$\begin{aligned} T_1 &:= E\text{-HA}^\omega + IP + AC + \Gamma \\ T_2 &:= WE\text{-HA}^\omega + IP_\forall + MP + AC + \Gamma_\forall \end{aligned}$$

By modified realisability as defined by G. Kreisel (21)  $T_1$  is interpretable in Gödel’s system  $T$ , that is  $T_1 \preceq T$ . On the other hand we have that Gödel’s functional interpretation (6) interprets  $T_2$  in  $T$  and so we have both  $T_1 \preceq T$  and  $T_2 \preceq T$ . Therefore we accept both  $T_1$  and  $T_2$  as truly constructive theories.<sup>13</sup> A crucial point here, which I will return to, is that these two theories are constructively incompatible.

An objection against this way of getting confidence in complex theories is found by asking: How do we know that the realisers actually do what is required? If we look at the details of the reductions then we see that this fact is actually proved using the exact principles in question. Using only basic mathematical reasoning we know for instance that if  $T_1 \vdash \exists x C(x)$ , for any formula  $C$  then there exists a term in  $T$  which we can find such that  $T_1 \vdash C(t)$ . The thing is that we really need the full theory  $T_1$  in order to prove that  $t$  has property  $C$ . However, we do not find this problematic. The whole thing is, first of all, computationally meaningful, and the properties that the theories possess are certainly not enjoyed by theories based on classical logic. But one could still argue that the theories incorporate principles which are non-intuitionistic and one may *therefore* be skeptical towards the theories. Now, such an attitude is dogmatic: Why should precisely intuitionistic logic monopolise constructivity?

<sup>13</sup>Note, however, that  $T_1$  is not closed under Markov’s *rule* as shown in (18).

The interpretations referred to here show that the respective principles are locally constructive—not that, say, Markov’s principle generally is constructive. But it is relative to typed arithmetic with weak extensionality together with a restricted form of the independence-of-premise and the axiom of choice. Note that Markov’s principle is not only validated for numbers (type  $o$ ) but is also validated in higher types. It is easy to attach some understanding to the principle for numbers, but this is surely not straight forward in higher types.

The really interesting thing is that many of these principles are in several *combinations* constructively problematic. As shown in (18) the combination of IP and MP is demonstratively non-constructive (relative to arithmetic) and also full extensionality together with MP is likewise problematic. In this case, if one wants to work with Markov’s principle and an extensional notion of equality, then one will have to work with weak extensionality only.

## 5 Conclusions

It is clear that there is no unique global characterisation of constructivity. Surely, the proof interpretation by Kolmogorov and Heyting can be seen as a global interpretation trying to characterise constructivity. But this should not be taken as more than a guideline or a rule of thumb. For as we have seen above, locally in the context of typed arithmetic one can constructively accept more. But then things turn out in a very complex manner: It is indeed a *very* subtle issue to combine the different principles which treated separately can be given an interpretation. One can obtain constructive methodologies in various different ways which are not in harmony with one and each other, since the constructivity of mathematical methods depends (also) on the context in which the methods are applied.

It is also clear that the finer details and the more information one seeks, the more complex and diverse things get. More information but also more diversity follow if one wants to do mathematics constructively. This extreme variation and diversity clearly causes pain among classical mathematicians and is, probably, a primary reason why doing constructive mathematics is not a popular occupation.

Here we will not try to answer why diversity occurs when one does constructive mathematics. But it is clear that when one tries to ascribe constructive meaning to different ideal elements then it gets really difficult to obtain coherent systems.

There are several conclusions to draw from these investigations. When one does mathematics and wants constructive results, an analysis of what is actually required seems to be a good idea. For a certain concrete purpose one chooses the most optimal system. If one tries to develop everything within one single system, then it is clear that many results cannot be obtained, since they may need principles which together with those accepted at the outset are non-constructive. Markov’s principle is, for instance, very useful—but maybe it is not such a good idea always to have it in your tool box.

Today larger and larger constructive systems are being build, but as the above has shown this is indeed a difficult task. First of all, one easily messes things up and gets non-constructive systems; second of all, one loses much information.

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