Kant and the natural numbers

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1 Introduction

The ontological status of mathematical objects is perhaps the most important unsolved problem in the philosophy of mathematics. It is thought-provoking that within the philosophy of mathematics there is no agreement on what the mathematical objects generally are. It is, moreover, surprising that not even the question of the ontology of the natural numbers is resolved, as these objects arguably are the most fundamental objects in mathematics.

In this paper I present a theory of the natural numbers which is based on the notion of *schema*. In fact the theory is based on a slight generalization of Kant's notion of objects and corresponding schemata. As such the theory is compatible with a kind of constructivism and the result is that numbers certainly do not belong to some transcendent Platonic realm. Contrary to the Platonic account the following theory presents numbers as a necessary element within *human* cognition. Numbers play a very essential role in our conceptualization of the world. As we shall see, a particular number is not characterized extensionally by objects representing the number. As we will will be working within a general constructivist epistemology the extension is—as we shall see below—not constant. On the other hand, the properties of numbers are independent of time; thus numbers are determined intensionally. The task of this paper is to state the details of this intensional aspect of the natural numbers. In Kant's theory of schemata this intensional aspect consist of number understood as a schema. Thus we will have to go back and see what Kant understood under this notion.[1]

[1] Kant's theory is very interesting in itself. It sheds light upon important topics such as the *objective* and the *subjective* and their interplay; it includes a theory about *mathematical reasoning* in general; it initiates a theory about *diagrammatic reasoning*; and it

In a short but unclear chapter entitled Schematism in the *Critique of Pure Reason* Kant outlined what he generally understood under the notion of Schema. Now, the first half of that book has three very important parts: The Deduction, the Schematism and the Principles. All three of them concern the relation between (pure) concepts and sensibility. Roughly it can be said, that the Deduction shows (or is intended to show) that there is harmony between the pure concepts of the understanding and the way appearances are given. This harmony ensures that appearances *can* be cognized under categories. The Schematism shows (or is intended to show) *how* the appearances are subsumed under categories. The Principles show (or are intended to show) what the general *conclusions* to be drawn are.

But the chapter on the Schematism concerns more generally the connection between the intellectual (the conceptual) and the physical (the sensible); it is not only about the schematism of the categories but about schematism of:

1. Empirical concepts,
2. Pure sensible (i.e., geometrical) concepts,
3. Pure concepts of understanding.

According to Kant, the categories can be divided into four different classes: The categories of i) quantity, ii) quality, iii) relation and iv) modality. In this essay I will concentrate mainly on the categories of quantity since the natural numbers are understood as a rule, namely the schema for the categories of quantity.

2 Number as schema

In the chapter on schematism we find the following description of the connection between size, number and schema:

> The pure **schema of magnitude** [***Größe***] (*quantitatis*), however, as a concept of the understanding, is **number,** which is a representation that summarizes [*zusammenfaßt*] the successive addition of one (homogeneous) unit to another. (A142/B182)

It must be admitted that even with regard to some of the most fundamental aspects of his epistemology Kant is not completely clear. Here, however, it seems clear that "magnitude [*Größe*]" is a category—but not which one. In the "Table of Categories" (A80/B106) there are four main divisions of the categories; the first one

anticipates the *type-token* distinction found in contemporary epistemology. Thus Kant's theory of schemata is interesting independently of his general theory of knowledge, i.e., his empirical realism and transcendental idealism. For an elaboration on these elements of schematism see my dissertation (Jørgensen, 2005).

being "Of Quantity [*Der Quantität*]". This consists of the three categories *unity, plurality* and *totality*. Now, "magnitude" could either refer to the collection of the three categories, or it could be one of them. Let us elaborate a little on this.

Quite generally Kant claims that it is *equivalent* to cognize and to perform judgments.[2] Therefore "the Clue to the Discovery of all Pure Concepts of the Understanding" (A70/B95) goes via the different ways we form judgments. In respect to quantity, Kant is completely Aristotelian: There are *singular, particular* and *universal* judgments (for instance, 'this body has mass', 'some bodies have mass' and 'all bodies have mass'). These different types of judgments lead Kant to the three categories unity, plurality and totality. The singular judgment corresponds to the category of unity; the particular judgment to plurality and the universal to totality.[3] Totally there are 12 categories, and just after stating what "the schema of magnitude" is Kant continues the Schematism by listing the nine remaining schemata—one after the other (A143–7/B182–7). This can be taken as evidence indicating that Kant took "the schema of magnitude" to be a schema common to the three categories unity, plurality and totality. And as Longuenesse notes (1998, p. 254), the three different categories seem to be involved in the definition of the schema: unity ("units"); plurality ("successive addition of one (homogeneous) unit to another"); and totality ("a representation that summarizes the successive addition of one (homogeneous) unit to another"). I think, however, that it is more plausible, to understand "the schema of magnitude" to be the schema of *totality*.[4] The third category in all the four divisions is always understood as a "combination of the first and second in order to bring forth the third" (B111). "Thus **allness** (totality) is nothing other than plurality considered as a unity" (B111). In *Prolegomena*, when

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[2] See for instance the important A69/B94:

> We can, however, trace all actions of the understanding back to judgments, so that the **understanding** in general can be represented as a **faculty for judging.** For according to what has been said above it is a faculty for thinking. Thinking is cognition through concepts. Concepts, however, as predicates of possible judgments, are related to some representation of a still undetermined object.

[3] It is remarkable that it is not even completely clear from Kant's texts precisely what the correspondence between the Quantity of Judgments and the Table of Categories is in this case. Just like Paton (1936, II, p. 44) and Longuenesse (1998, p. 249), I understand Kant in such a way that the (Aristotelian) judgments are given in the traditional order: Universal–particular–singular, whereas the order of the categories are given as unity–plurality–universal. Therefore, the order of one of the tables is the reverse of the other. This interpretation is in contrast with Hartnack (1968, p. 39) and (Tiles, 2004). Tiles has a refined argument for the opposite of the view expressed here.

[4] Here I am close to Longuenesse (1998, pp. 253–255), although she seems to *identify* plurality and totality.

listing the categories, Kant terms the first category of Quantity as "Unity (Measure [*das Maß*])" (Ak. 4, p. 303).[5] Therefore, *Unity* is the concept we use when we judge something to be a unit; 'this body—taken as a unit—has mass'; but also 'this unit can be taken as a unit, when we want to count—it can be seen as (giving rise to) a measure'. A determination of a unit 'body' together with a judgment 'here is a plurality of divisible bodies' can form a new judgment, when we think another unit: 'All bodies in *this collection* are divisible'.

> [T]o bring forth the third [pure] concept requires a special act of the understanding, which is not identical with that act performed in the first and second. Thus the concept of a **number** (which belongs to the category of allness) is not always possible wherever the concepts of multitude and of unity are (e.g., in the representation of the infinite) (B111).

Therefore, the category of totality is not reducible to unity and plurality and, moreover, it is "number" which belongs to totality. If we re-read the definition of "the schema of magnitude" in this light we see that it corresponds very well to the act performed, when totality—as a combination of unity and plurality—is performed: A certain unit (measure) has been determined; and we have encountered a plurality of these units. When reflecting *on* this plurality, we form a set out of the homogeneous elements and *enumerate* (summarize) the elements. This enumeration ends with a number, which is the number of elements in the set.[6] Therefore, when we apply "the schema of magnitude" we think unity in a given plurality. Generally "the schema of magnitude" is the ability humans have for determining finite extensions of empirical concepts. For instance, 'there are five fingers on my

[5] I use (Ak. *volume*, p. *page*) when I refer to *Kants Gesammelte Werke* in the so-called Akademie version.

[6] But as Kant notes, we cannot reduce the third category to the two foregoing, because sometimes we encounter an infinite plurality of units which do not form a set. In this way, the set-class distinction from set theory is reflected in Kant's epistemology. We can for instance, form a sequence of growing finite (in the sense that the volume is finite) empirical spaces. The space containing my office; the space containing my city; the space containing my country, the earth and so on. All these spaces are possible objects of intuition. But the *collection* of all finite space—the absolute empirical space—which is needed for Newtonian mechanics in the definition of absolute motion is not a possible object of intuition and "absolute space is *in itself* therefore nothing and indeed no object at all" as Kant writes in *Metaphysical Foundation* (Ak. 4, p. 481). I can measure all the finite spaces, but their collection is not measurable. Nevertheless, the concept of "*motion in absolute* (immovable) *space*" "in general natural science is unavoidable" (Ak. 4, p. 558), so "[a]bsolute space is therefore necessary, not as a concept of an actual object, but rather as an idea", i.e., as a concept of *reason* (Ak. 4, p. 560). In set theory any *set* of ordinals is itself measurable by an ordinal, but the class of all ordinals, is not a set, because it is not measurable by any ordinal.

left hand': The unit is 'finger'; the context is 'my left hand'; and there are five of them in total. The rule determining this act of enumerating and counting is the schema called "number".

But how precisely do we operate with this schema? It is a transcendental schema and there are certain important differences between the transcendental schemata and the geometrical schemata. "The schema of a pure concept of the understanding, on the contrary [to empirical and geometrical schemata], is something that can never be brought to an image at all" (A142/B181).[7] Rather than producing images, transcendental schemata provide a "transcendental time-determination" (A138–9/B178–9) of the objects given in experience. As empirical objects are given to us in outer sense (space) and time-determination is a determination of moments according to inner sense,[8] this determination proceeds *mediately* by way of inner images.[9] Thus the schematism of pure concepts—in contrast to geometrical and empirical concepts—is more about *reflection* on images, rather than *construction*. This observation somehow runs counter to Shabel's (2003, p. 109) claim that the diagrammatic reasoning in Euclid—which Kant supplies an epistemological analysis of in terms of geometrical schemata—"provides an interpretive model for the function of a transcendental schema". Nevertheless, it will become clear that Kant's theory of geometrical schemata and his theory of transcendental schemata have many properties in common. The most important one being that schemata provide a foundation and explanation of the use of types and tokens.

In the following I will give my interpretation of the transcendental "**schema of magnitude**". As will be clear this really is an interpretation. Kant does not write much about the transcendental schemata and many of the details are 'left to the reader as an easy exercise'. But this exercise is not an easy one, as Kant certainly is not very clear on the whole issue. This is in contrast with the geometrical schemata about which Kant's theory is much clearer. In consequence of this, I will in the first place not stay close to the text. This is for the sake of giving a coherent (as coherent as it can be, at least) interpretation of the schema of magnitude. After having provided my interpretation I will go back to Kant's text and see whether or

[7] In fact, this fundamental difference between the arithmetical schema number and the geometrical schemata leads Kant to reject the possibility axioms for arithmetic. This well-known claim of Kant is expressed on pages A163–4/B205. This view seems rather awkward in a contemporary understanding and I will elaborate on it below.

[8] "Time is nothing other than the form of inner sense, i.e., of the intuition of our self and our inner state. For time cannot be a determination of outer appearances; it belongs neither to a shape or a position, etc., but on the contrary determines the relation of representations in our inner state." (A33/B49-50)

[9] "[T]ime is an *a priori* condition of all appearance in general, and indeed the immediate condition of the inner intuition (of our souls), and thereby also the mediate condition of outer appearances." (A34/B51)

not I am able to supply difficult passages with meaning. I think my interpretation should be more than just compatible with what Kant originally wrote. It should also shed new light on difficult passages. If I succeed in doing this my interpretation will be justified.

Let me return for a moment to the empirical schemata which I will give the following interpretation. Let us view the collection of all the different images representing empirical objects as a constructive and non-monotone open-ended universe. It is not monotonic as our empirical concepts may vary over time and it is constructive in the sense that we produce images in time in accordance with rules. Furthermore, it is open-ended as our collection of empirical concepts is certainly not fixed once and for all. However, given a point i in time we can take the universe of images which are *in principle constructible* according to the collection of empirical concepts we may possess at i. Let us call this snapshot U_i. Any empirical concept partitions U_i in two sets. One of the sets consisting of the images representing the concept, the other set consisting of those which do not represent the concept.[10]

"[M]agnitude (*quantitatis*)" concerns the question: "How big is something?" (A163/B204). An example could be: 'How many fingers are there on my left hand?'. Let $\prec$ denote the order of time.[11] I have experienced a plurality of fingers and let $i \prec j \prec k \prec l \prec m$ be the different moments in time corresponding to these experiences. Although the sequence of image-universes is generally not monotone, let me assume that *locally* there is monotonicity such that the partitions of $U_i, \ldots, U_m$ consisting of finger-images are the same from i to m. Alternatively we could assume that the schema belonging to the concept 'finger-on-my-left-hand' is constant. Such a pragmatic assumption seems reasonable.[12] Therefore *any* image from the partition can be used to represent any finger from time i to time m. Let x be such an image. x represents any of the fingers that I experience on my left hand. The minimal requirement making the images different is their location in time.[13]

[10] Of course this is an idealization which is perhaps not fully justified. Empirical concepts are vague concepts, and therefore it is perhaps not possible, given a concept P, to form two *disjoint* sets U_i^P and $U_i^{\neg P}$ such that $U_i = U_i^P \cup U_i^{\neg P}$. This problem is, however, not a problem which threatens the interpretation of "the schema of magnitude", and therefore I will make this idealization.

[11] Kant's precise understanding of the order of time is not important for this example. For our example it only matters that the collection of past moments is linearly ordered.

[12] The assumption of monotonicity locally around a concept seems reasonable, given that I in the time from i to m do not discover new essential properties of the concept, and given that I do not forget any of the essential properties either.

[13] We could also imagine another way of making the images different, namely that they are cognized as fingers with coordinates in space. This would also separate the images. But this would only be an additional property making the images different, because we can-

Time “determines the relation of representations in our inner state” (A33/B49-50). Therefore we have *temporized* images: x_i, x_j, x_k, x_l and x_m. This is my interpretation of Kant’s “transcendental time-determination”, and note that the temporal indexes are *necessary* for my judgment ‘there is a plurality of fingers’. Now, by judging unity in plurality I form the set M consisting of the temporized images which I simultaneously count (“summarize”). This is done by enumerating M. Mathematically speaking this is a determination of a set of natural numbers which is equinumerous with M. In other words, we determine a set of natural numbers N and establish an injection f from N to M. The canonical domain for f is, of course, $\{1,2,3,4,5\}$. Figure 1 represents this mental process.

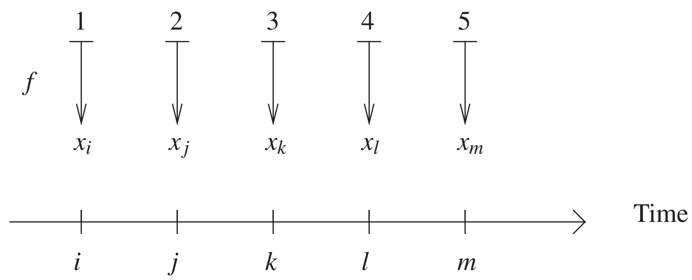

Fig. 1.

On pages A142–3/B182 Kant defines the “schema of magnitude” in one paragraph. I quoted gave most of the first half of that paragraph on page 2. The second part goes:

> Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition. (A142–3/B182)

In terms of the finger-example my interpretation is the following:

Firstly, I can produce the injection f “because I generate time itself in the apprehension of the intuition”—it is I who locate the images according to inner

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not help that the images are *constructed* in time. Thus the images with spatial coordinates would be different both with respect to time and coordinates.

sense. This temporization is necessary. Secondly, that "number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general" means that the number 5 describes the unity of my set of fingers in the sense that a bijection (f is of course also surjective) can be established between the set of natural numbers less than or equal to 5 and M. But what precisely does the correspondence between 5 and M consist of?

The order of time $\prec$ induces a natural order on the temporized images; let $\prec$ denote also the induced order. The function f specified in Figure 1 is actually showing that $(\{1,2,3,4,5\},<)$ and $(M,\prec)$ are isomorphic. Could it be a notion along these lines that Kant has in mind when saying that number 5 *is* "the unity of the synthesis"? The bijection established would then show that the two sets are *equal* up to isomorphism. I do not think this is Kant's idea. We are only interested in the *size (Größe)* of M, not its order. It is therefore 5 as a cardinal number and not as an ordinal number Kant is interested in. The unity Kant is after is the unity which is expressed through *cardinality*. A contemporary mathematical understanding of size is the following. A and B are *equinumerous* or *equal in cardinality*, if a bijection between the two exists:

$$A =_c B, \text{ if and only if, there exists a bijection } f \text{ such that } f : A \to B.$$

Of course, given a set N of temporized images with n elements any bijection g between N and $\{1,\ldots,n\}$ gives rise to an isomorphism between $(\{1,\ldots,n\},<)$ and $(N,<_g)$, where $<_g$ is the order induced by g—but this additional information is not what Kant has in mind. It is the information which lies *in the process* determining the cardinality of any finite set, but one has to pay attention to this. I understand Kant as saying that this type of information is not what we are after when applying the category of totality, although it can be unwinded from the process. Thus, in the finger example 5 reflects the unity of the synthesis with respect to *cardinality*. The relation of equality is not isomorphism, rather it is $=_c$. The function f depicted in Figure 1 could consequently be any bijection between M and $\{1,2,3,4,5\}$.[14]

I can now give my general interpretation of the "schema of magnitude". The task of the schema is to give unity to a succession of objects thought under the same concept; objects that we represent as temporized images. In our interaction with the world we temporize images. Before temporization the images belong to the universe of images constructible in principle. For the sake of presentational simplicity let us assume that this universe is constant, i.e., that my empirical schemata generating images are the same over time. Later I will dismiss this restriction. From this universe we form, by "transcendental time-determination" a temporized derivative, namely, the universe U of temporized images.

The general problem concerns the possible unity of a plurality of experiences of objects falling under a concept. Let us in accordance with this purpose form a

[14] Of which there are many: 5!.

second-order universe $\mathscr{U}$ consisting of sets of temporized images representing the same concept. Thus the definition of $\mathscr{U}$:

$M \in \mathscr{U}$ iff M is a subset of U and the members of M represent the same concept.

Clearly the notion of *equal in cardinality*, $=_c$ is an equivalence relation on $\mathscr{U}$. The "schema of magnitude", which Kant calls "number", is the general rule which generates this equivalence relation. When we judge unity in a plurality of homogeneous units we determine the equivalence class to which the unit belongs. And if no such class can be found we cannot judge unity: A plurality of experienced units can be judged to be a unit itself *only if* we can determine a cardinality of this plurality. Therefore, the generation of the members of $\mathscr{U}$ proceeds by production of the equivalence classes. An M becomes a member of $\mathscr{U}$, when it becomes a member of an equivalence class of $\mathscr{U}$. Consequently, it is a genuine *constructive* notion of existence.

Let us now see, what the content of the rule "number" is. First of all, it consists of a "transcendental time-determination", as otherwise we cannot distinguish between images representing the same concept. But what is the content of the act which partitions $\mathscr{U}$ in accordance with $=_c$? It is our ability to enumerate finite sets, and the mathematical description of this is the ability to produce bijections between finite sets.

In this way, we see that the "schema of magnitude" is reflective because of reflection on already constructed first-order level images. But it is constructive on a second-order level; the sets of $\mathscr{U}$ are determined (second-order constructed) by the schema, and they live in $\mathscr{U}$ only when this determination has taken place. Moreover, the schema is an *act* of the understanding which itself takes place in time, temporal succession is a transcendental condition:

> No one can define the concept of magnitude in general except by something like this: That it is the determination of a thing through which it can be thought how many units are posited in it. Only this how–many–times is grounded on successive repetition, thus on time and the synthesis (of the homogeneous) in it. (A242/B300)

Let us now dismiss the restriction posed on the universe of images. Thus a certain invariance of meaning of images can happen, as an image can represent a concept at a certain time and not represent that concept at another time. As a consequence of this the universe $\mathscr{U}$ is a dynamical floating universe where the elements of the equivalence classes defined by $=_c$ are not the same over time. Does this pose a problem for the status of the *pure* concept of totality? Certainly not, given any variant of $\mathscr{U}$ the rules determining equivalence classes according to $=_c$ are precisely the same. The underlying universe may vary, as our empirical concepts vary, but the structure imposed on it is the same. Of course there are extensions of empirical concepts which we are unable to determine, of which the sorites paradox provides an excellent example. But this is not a problem for our theory of magnitude, it is (perhaps) a problem for our theory of empirical concepts.

3 Number as concept

It must be admitted that my interpretation is somewhat involved. I am, for instance, using notions like equivalence-class and second order object—notions which were not present in the mathematics of Kant's time. On the other hand, it is also true that the bijections involved are constructively fully meaningful: We are only working with functions from finite sets to finite sets—the essence of that is simply to pair objects from two different finite sets. Moreover, the universes mentioned should certainly *not* be understood in some kind of Platonic sense—rather they are universes constructed by a cognizing human.

Some interpretation *is* needed as Kant's own text is unclear and lacks important details. Kant designates the constructive procedure used when counting as "**number**".[15] But from his text (or texts) alone it is not clear precisely what he means. For a full justification of my interpretation I should be able, however, to explain central themes in the Kantian theory of transcendental schemata. One of the most distinctive ones is, that arithmetic has no axioms. I will take that up in the last section. Another important distinction in Kant's theory is the distinction between 'number as concept' as opposed to 'number as schema'.

Kant notes on A142/B181 that a transcendental schema "is something that can never be brought to an image at all". His point is that there are no generic images of the pure concepts. In contrast to empirical schemata, pure schemata do not provide images representing the pure concept. How would a paradigmatic image of causality look, for instance?[16] Somehow in contrast to this Kant writes, when discussing the difference between image and schema, that

> if I place five points in a row, ..., this is an image of the number five. On the contrary, if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself (A149/B179)

There is, therefore, a concept for the number five, but there is also a schema called "number" which is an act of the understanding or a "representation of a method

[15] Note also that in German to count is *zählen*—a derivative of *Zahl*.

[16] One who was looking for a paradigmatic image of causality was Hume. According to Kant this cannot be found.

for representing a multitude". Therefore we have to distinguish between the concept number and the schema number. The number five is *not* the rule synthesizing my finger-images x_i, x_j, x_k, x_l, x_m, but rather a concept of the specific size—the cardinality—of the corresponding set. This cardinality is realized through the enumeration which simultaneously determines set-hood (unity) and cardinality. Thus number is a concept under which a multiplicity is thought. This concept is thought through the schema "number". The schema is the *procedure*, i.e., a rule-governed activity, that we use to determine whether a given collection of sensible things exhibits unity in plurality.[17]

Let $\overline{n}$ denote the class of elements in $\mathscr{U}$ which are equivalent $\{1,\ldots,n\}$ with respect to cardinality, in other words:

$$\overline{n} = \{\,\{x_1,\ldots,x_n\} \in \mathscr{U} \mid \{x_1,\ldots,x_n\} =_c \{1,\ldots,n\}\,\}.$$

Now, I propose to understand the *concept* of a particular number *n* as a type, in fact more specifically as the corresponding equivalence class $\overline{n}$. We know, however, that the elements of this equivalence class is not constant: An element of $\mathscr{U}$ is constructed as a second-order object at a certain moment in time. Through this construction the element becomes a member of an equivalence class. Therefore, if we understand the concept five as an equivalence class in the set theoretic sense described above, then the equivalence class is *not* determined by its extension, it is rather determined intensionally, namely the "schema of magnitude" which amounts to the capacity of producing bijections. Only this *intensional* aspect can guarantee that number concepts remain the same over time. Therefore, if the $\overline{x}$ and $\overline{y}$ are number concepts, possibly 'found' at different moments in time, then the equality of $\overline{x}$ and $\overline{y}$ is determined not by their extensions but by the bijections—understood as rules—on which they are generated. In other words $\overline{x} = \overline{y}$, if and only if, the two canonical bijections are the same.

We understand a concept of a certain number as a cardinal number being a *type* whose tokens are members of the corresponding equivalence class. The "schema of magnitude" decides the relation between the type $\overline{n}$ and the tokens falling under this type. It is due to the intensional aspect that the properties a particular number concept are independent of time. Moreover, each and all of the (finite) cardinal numbers are founded by the same schema. See Figure 2 for a diagram represent-

[17] In the A-deduction Kant writes: " If, in counting, I forget that the units that how hover before my senses were successively added to each other by me, then I would not cognize the generation of the multitude through this successive addition of one to the other, and consequently I would not cognize the number; for this concept consist solely in the consciousness of this unity of the synthesis" (A103).

ing my interpretation.[18] So the schema and the concept of number certainly are different.

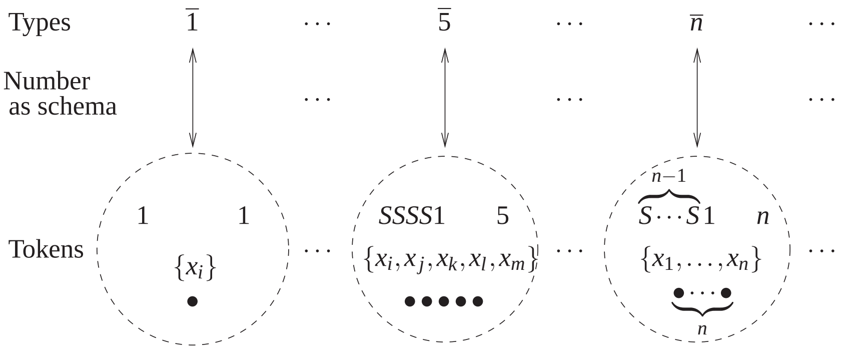

Fig. 2.

4 The arithmetical schema and universality

In the case of geometry, geometrical schemata make reasoning about say the general triangle possible through reasoning about one particular.[19] My claim is that Kant holds the view that we are in a similar position in the case of arithmetic. The form of universality which emerges in the case of arithmetical schemata can be gained by using "the schema of magnitude". On the face of it, this may seem problematic as the extension of an arithmetical equivalence class is not constant. On the other hand, the members of the geometrical types are pure intuitions and a pure intuition belongs to the type triangle, if and only if, it can be constructed in pure intuition by the schema of triangle. This property is invariant over time—therefore the geometrical types are extensionally determined. Nevertheless, when explaining the synthetic nature of propositions in arithmetic Kant writes:

> The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze

[18] Although generally my interpretation is different from the one given by Longuenesse (1998) I think my understanding of the difference between the concept number and the schema is close to her's, according to which "the concepts of number is the concept of a *determinate* quantity, and that number as a schema is the schema of *determinate* quantity" (1998, p. 256).

[19] See for instance (Jørgensen, 2005, pp. 23).

> my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B15–6)

This quote is central for an understanding of Kant's conception of numbers and arithmetic. Kant is claiming two important properties of arithmetic:

1. The simple propositions of arithmetic like $7+5=12$ are not analytic; "[o]ne must go beyond these concepts, seeking assistance in the intuition" in order to realize that it actually is the case that 5 added to 7 yields 12.
2. For the verification of the correctness of $7+5=12$ I can use contingent *empirical* representations of 5—fingers on my hand.

I will treat the syntheticity of numbers in the next section. Here I focus on the second point. How can I obtain a necessary proposition about all possible sets with cardinality 5 by using one particular set with cardinality 5? It could happen that I by accident could loose a finger, or that my concept of "finger-on-my-hand" would change over time such that a thumb would no longer be a finger. The answer is of course that due to the "schema of magnitude" we realize the following: When we use the set of (temporized images of the) five fingers we use it *only* with respect to its magnitude: We understand that we could have used any other set with five members, i.e., we could have used any other member of the equivalence class $\overline{5}$ for the verification. The situation is strikingly close to geometrical schemata; only there, however, it is about construction and not reflection:

> The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. (A713–4/B741–2)

In the case of the set of five finger-images, we realize they represent "the concept without damage to its universality, for in the case of this empirical intuition we have taken account only" of its size, i.e., its cardinality, as determined by the "schema of magnitude". The construction which takes place is on a higher level, namely that *we think* of the five fingers as constituting a set. We realize, however, that we could have used any other set with this magnitude, and the similarity to the geometrical reasoning is striking: When we prove a proposition about a concept, say the number five or a triangle, we take an intuition representing the concept *together* with the rules determining any intuition falling under that concept. The "schema of magnitude" thus ensures that we can operate with specific images of numbers, taking them as representatives of their types, prove properties about the images and be sure that these properties apply, not only to the specific images (i.e., the tokens) but to any image representing the type.

Precisely this aspect of Kant's schematism was simply not understood by G. Frege (1848–1925) in *The Foundations of Arithmetic* (1884/1980). In § 13 Frege ascribes to Kant the position that "each number has its own peculiarities. To what extent a given particular number can represent all the others, and at what point its own special character comes into play, cannot be laid down generally in advance." Well, the schema will take care of this, according to Kant.

It should be noted that in contrast to geometry it is not by first-order construction rather it is by reflection the determination of cardinality takes place. By this I mean the following. The construction which takes place in geometry is construction of pure intuitions which are first-order objects belonging to U. On the other hand, arithmetical reflection is construction of second-order objects, subsets of images, belonging to $\mathscr{U}$.

5 Syntheticity of arithmetical propositions: Space and time

Both space and time ground our notion of number. Kant's point when saying "[t]houghts without content are empty"[20] (A51/B75) is precisely that if, say the equivalence class corresponding to some number is empty, then thinking about that particular number is not possible. The elements of the equivalence classes are mental objects, of which there are two kinds: pure intuitions and images referring to empirical objects. But in either case they refer mediately or immediately to spatial objects. This relation between concepts and intuitions is exemplified in the relation between numbers as types and equivalence classes, by saying "[o]ne must go beyond these concepts, seeking assistance in the intuition" (B15). Pure intuitions, however, gain objectivity through and only through the empirical:

> Now the object cannot be given to a concept otherwise than in intuition, and, even if a pure intuition is possible *a priori* prior to the object, then even this can acquire its object, thus its objective validity only through empirical intuition, of which it is the mere form. Thus all concepts and with them all principles, however *a priori* they may be, are nevertheless related

[20] Which means 'concepts without intuitions'.

> to empirical intuitions, i.e., to *data* for possible experience. Without this they have no objective validity at all, but are rather a mere play, whether it be with representations of the imagination or of the understanding. One need only take as an example the concepts of mathematics, and first, indeed, in their pure intuitions. Space has three dimensions, between two points there can be only one straight line, etc. Although all these principles, and the representation of the object with which this science occupies itself, are generated in the mind completely *a priori*, they would still not signify anything at all if we could not always exhibit their significance in appearances (empirical objects). [. . .] In the same science [mathematics] the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes. The concept is always generated *a priori*, together with the synthetic principles or formulas from such concepts; but their use and relation to supposed objects can in the end be sought nowhere but in experience, the possibility of which (as far as its form is concerned) is contained in them *a priori*. (A239–40/B298–9)

If the proposition $7+5=12$ had been analytic then merely thinking about the concepts of $5,7$ and $+$ would yield the result 12. But this is not possible. Our concept of 5 is not a collection of marks, rather it is an abstract type, whose semantics is determined by the "schema of magnitude" when the subject is interacting with the world, and therefore numbers have meaning only in connection with intuitions. We have to operate with particular tokens which are found "in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes". A consequence of this is that the notion of number is meaningless only if all equivalence classes in $\mathscr{U}$ are empty. In other words, our notion of number is meaningless only if no objects are representable. Thus it becomes an issue to guarantee that these classes are not empty, and this seems to lead to a problem about large numbers. As Frege puts it:

> I must protest against the generality of Kant's dictum: without sensibility no object would be given to us. [. . .] Even those who hold that the smaller numbers are intuitable, must at least concede that they cannot be given in intuition any of the numbers greater that $1000^{1000^{1000}}$, about which nevertheless we have plenty of information. (Frege, 1884/1980, p. 101)

Tait formulates in the paper *Finitism* a similar criticism, which is extended to a critique of Kant's notion of number through a critique of Kant's dictum (in the words of Tait) that "existence is restricted to what can be represented in intuition" (2005, p. 7). Tait writes:

> It is clear—and was so to Kant and Hilbert—that there are numbers, say 10^{10} or 30, which are not in any reasonable sense representable in intuition. Kant seems to have responded to this by saying that at least their parts are representable in intuition [...] The real difficulty, however, is that the essence of the idea of Number is iteration. However and in whatever sense one can represent the operation of successor, to understand Number one must understand the idea of iterating this operation. But to have this idea, itself not found in intuition, is to have the idea of number *independent of any sort of representation*. (Tait, 2005, p. 35)

In should be clear from my exposition of Kant's notion of number that Kant is not in any way affected by the latter part of Tait's criticism. Kant does not hold an empirical understanding of number—rather the concept of a general number is founded on the "schema of magnitude" which flows from the intellect, certainly not from the empirical.

In the former part of the criticism Tait seems to have the same premise as does Frege. Intuition, on their understanding, does not include pure intuition. Granting that we have *pure intuition* Kant would respond by saying that by *the very writing* of $1000^{1000^{1000}}$ or 10^{10} or 30 we in fact *have* intuitions. The very inscriptions provided by Frege and Tait are intuitions. They are understood in terms of the exponentiation function which is basically a primitive recursive function. Therefore, in principle, we can determine the equivalence-class-membership of these inscriptions—they are numbers, as we have an intuition and a rule determining how to operate with this intuition.

Let me give a general solution concerning the meaningfulness of large numbers. As it turns out it is our concept of space which ultimately provides arithmetic with its objects. Our primary geometrical schemata (some equivalents of Euclid's postulates) lead to a production of a sequence of finite spaces,

$$S_1, S_2, \ldots, S_n, \ldots$$

where S_i is strictly smaller than S_j, if $i < j$, and the whole sequence is unbounded. This sequence of pure spaces is a constructive but potentially infinite sequence. Thus given any natural number the equivalence class corresponding to that number is inhabited, at least due to this sequence of increasing finite pure spaces. Therefore, the justification of this argument, which refutes Frege's criticism, rests on Kant's notion of space (see my analysis in Jørgensen, 2005).

Time, however, is also a necessary condition for the concept of magnitude. The concept of iteration is a necessary element in the "schema of magnitude" ("the successive addition of one (homogeneous) unit to another"). Without iteration it would be impossible to determine the magnitude for any given thing. Kant assumes nothing particular about the objects for numbers—they can be anything—but adding unit to unit always takes place in time: “this how–many–times is grounded on successive repetition, thus on time” (A242). Therefore, the natural numbers as a sequence of numbers can only be *represented* as a progression in time. Furthermore, also the most simple operations of arithmetic, say addition, takes place in time according to Kant. In a letter to Schultz Kant writes: “If I view $3+4$ as the expression of a *problem*” then the results found “through the successive addition that brings forth the number 4, only set into operation as a continuation of the enumeration of the number 3” (Ak. 10, p. 556). So I can take first three fingers together with four fingers and enumerate all of them. This enumeration ends by judging the fingers to constitute a set with cardinality 7.[21]

Under the condition of inner sense (time), number as *schema* generates the synthesis (of representations of) objects subsumed under a concept. As a consequence of this cognition, number, pure concept, representations of objects falling under an empirical concept, transcendental time-determination and the transcendental imagination are closely related *in inner sense*, as also Figure 1 illustrates: It all takes place under the conditions of inner sense.

But in the same letter to Schultz, Kant claims that:

> Time, as you correctly notice, has no influence on the properties of numbers (as pure determinations of magnitude) [...] The science of numbers, notwithstanding the succession that every construct of magnitude requires, is a purely intellectual synthesis, which we represent to ourselves in thought. But insofar as specific magnitudes (*quanta*) are to be determined in accordance with this science; and this grasping must be subjected to the condition of time. (Ak. 10, pp. 556–57)

The natural numbers are pure concepts (types) of the understanding (*Verstand*), in the sense that they are not derived from experience, but from the structure of our representation.[22] Time has no influence on these types, as they are not dependent on time. The rules determining the tokens (members of the equivalence classes) are purely intellectual rules, which remain the same over time. This is the intensional aspect of number. The objects in the equivalence classes, on the other hand, are ultimately empirical objects (and therefore spatial). But reasoning about numbers proceeds necessarily by way of mental images over time. When we do mathematics and examine the properties of numbers, time is a necessary condition for the

[21] See also A164/B205.

[22] It is of course an interpretation to say that the numbers are pure concepts of the understanding, as Kant claims there are only 12 of these categories. It would, however, not make much sense, I think, to regard the numbers as anything but pure. In this respect I completely agree with M. Young (1992, p. 174), and I am generally sympathetic towards his short interpretation of Kant's schematism.

representations of numbers.[23] The interplay between pure concepts, tokens, time and space is summarized by Kant in Dissertation by saying that

> there is a certain concept which itself, indeed belongs to the understanding but of which the actualization in the concrete (*actuatio in concreto*) requires the auxiliary notions of time and space (by successively adding a number of things and setting them simultaneously side by side). This is the concept of *number*, which is the concept treated in ARITHMETIC. (Ak. 2, p. 397)

Therefore both space and time condition our access to and the constitution of the natural numbers: They are constituted by non-temporal schemata but any use will be temporal, and their meaning is ultimately provided by intuitions.

6 What numbers are

Numbers are not characterized extensionally—this would not be meaningful in Kant's framework. Number is rather given an *intensional* characterization in terms a collection of rules. In order to fully appreciate this and to give a coherent account we need Kant's full theory of schemata—all the way through empirical, geometrical and transcendental schemata. On the other hand, we really get a coherent interpretation when the disentangled theory of schemata is taken into consideration. I think that Charles Parsons failed to realize this when writing his article "Arithmetic and the Categories". There he concludes that "Kant did not reach a stable position on the place of the concept of number in relation to the categories and the forms of intuition" (1992, p. 152). In contrast to this, I hope my interpretation of Kant's theory of schemata has shown, precisely how Kant's notion of number relates to the categories, and to the two forms of intuition.

But, according to Kant numbers cannot be objects, as arithmetic has no axioms.

> The self-evident propositions of numerical relation [. . .] are to be sure, synthetic, but not general, like those of geometry, and for that reason also cannot be called axioms, but could rather be named numerical formulas. (A164/B205)

The numerical formulae Kant is thinking of are propositions like "$7+5=12$". But "[s]uch propositions must [. . .] not be called axioms (for otherwise there would be infinitely many of them)" (A165/B205). In a sense Kant is very right: *There was no axiomatization of number theory at the time of Kant*. In fact Kant reflects, once

[23] Therefore, the situation is not as in mechanics, where time is analyzed together with the alteration of placement in space.

again, the Euclidean paradigm. Euclid has no axioms for numbers. Number theory is treated in Book VII, and already in proposition 2 (the greatest common divisor of two numbers) Euclid uses well-ordering of the natural numbers, but without reference to some first principle.

Now, one could speculate: If Kant had had axioms for number theory, would he have regarded them as genuine axioms? Axioms should be synthetic a priori in analogy to the way that Euclid's postulates reflect schematic spatial procedures.

According to Kant geometrical schemata produce true objects in pure intuition. These are genuine objects in the sense that they are possible objects of experience. Let us use the terminology of a first-order universe U and a second-order universe $\mathscr{U}$, which I introduced on page 9. The objects which the geometrical schemata produce are elements of U. Representations of numbers, however, are elements in the second-order universe $\mathscr{U}$. Thus, they are not really objects to Kant as the only true objects are first-order objects. This also explains why the pure "schema of magnitude" deals more with reflection than construction. The elements of $\mathscr{U}$ are according to Kant not really constructed, they rather reflect a certain relation between objects living in U. I think this is the only good reason Kant has when claiming that the numbers are not objects.

In the course of history we have learned—due to relativity theory—that space is only *approximately* Euclidean. Consequently it seems that the objects we produce in Euclidean geometry are only approximations of possible objects of experience. On the other hand, the number five is still represented by the set of fingers on a normal hand. If we allow second-order objects to be genuine objects, then in fact they are more objective (in the Kantian sense) than the objects of Euclidean geometry.

Today we can give an axiomatization of the natural numbers, which relative to the slight generalization of the Kantian notion of object, is a set of axioms in the Kantian sense.

We can define an arithmetic Q, called Robinson arithmetic, in which there is a constant 0, called zero, a unary operation S, called successor and two binary operations $+$ and $\cdot$ called plus and times, which satisfy the following axioms:

1. $\forall x(0 \neq Sx)$.
2. $\forall x,y\big((Sx = Sy) \to (x = y)\big)$.
3. $\forall x\big(0 \neq x \to \exists y(x = Sy)\big)$.
4. $\forall x(x + 0 = x)$.
5. $\forall x,y\big(x + Sy = S(x + y)\big)$.
6. $\forall x(x \cdot 0 = 0)$.
7. $\forall x,y\big(x \cdot Sy = (x \cdot y) + x\big)$.

The first three axioms are realized by the "schema of magnitude". They claim the existence of a concept of iteration S which taken together with a symbol 0 gives rise to a paradigmatic representation of the natural numbers:

$$0,\ S0,\ SS0,\ SSS0, \ldots$$

Due to our ability of producing bijections, these paradigmatic numbers as intuitions put us in a situation where we can use and reason universally about *any* representation of the numbers. Recall, the schema together with any representation, whether empirical or pure, allow for universal reasoning. The sequence is furthermore potentially infinite, and thus for any type $\overline{n}$, the number $\overline{n}$ is meaningful. The realization of the latter axioms is also due to our ability to produce and operate with bijections. Plus corresponds to composition of functions, which we found on page 13 was validated by the "schema of magnitude", and times iterates this concept.

If we therefore allow second-order objects to be objects and accept the above axiom system, then there are numbers, just as much as there are triangles—or perhaps even more. We meet them as tokens and any number $\overline{n}$ is schematisable in the sense that we can produce a representation $S \cdots S0$ such that the representation together with the schema allows for universal reasoning. Finally, the interpretation I have suggested also makes certain elements found the first *Critique* more coherent, as for instance when Kant writes about number images (A149/B179; A240/B299).

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