Chapter 2
From the Practice of Explanation to the Ideology of Demonstration: An Informal Essay

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Abstract The following discusses the practice of mathematical argument or demonstration—at first based on what I shall speak of as "the locally obvious", that is, presuppositions which the interlocutor—or, in case of writing, the imagined or "model" reader—will accept as obvious; next in its interaction with *critique*, investigation of the conditions for the validity of the seemingly obvious as well as the limits of this validity. This is done, in part through analysis of material produced within late medieval Italian *abbacus* culture, in part from a perspective offered by the Old Babylonian mathematical corpus—both sufficiently distant from what we are familiar with to make phenomena visible which in our daily life go as unnoticed as the air we breathe; that is, they allow *Verfremdung*. These tools are then applied to the development from argued procedure toward axiomatics in ancient Greece, from the mid-fifth to the mid-third century BCE. Finally is discussed the further development of ancient demonstrative mathematics, when axiomatization, at first a practice, then a norm, in the end became an ideology. The whole is rounded off by a few polemical remarks about present-day beliefs concerning the character of mathematics.

Keywords Dardi of Pisa · Explanation as demonstration · Old Babylonian mathematical critique · Critique in early Greek geometry · Hippocrates of Chios · Oinopides · Euclid · Simplicios

1 Arguing from the Locally Obvious

Let us start with this piece from Dardi of Pisa's *Aliabraa argibra*, written in 1344, presumably in Veneto.[1] It comes from the first part of the treatise, which teaches the arithmetic of monomials, binomials and polynomials containing radicals.

[1] When at home in Pisa, Dardi would obviously not be identified as coming from there. Where then did he write? The oldest manuscript (Vatican, Chigi M.VIII.170) is written in Venetian, which does

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The passage teaches how to divide *number* by *number plus square root,* and is based on the example $\frac{8}{3+\sqrt{4}}$. I transcribe in modern notation—Dardi has *più* where I write "+", *meno* where I have "–", and ℞ where I use $\sqrt{}$. Finally, I write the division as a fraction—this is less innocuous but useful for our later argument.[2]

Building on what he has already taught, Dardi starts by calculating that $\left(3+\sqrt{4}\right)\cdot\left(3-\sqrt{4}\right)=3^2-\left(\sqrt{4}\right)^2=5$. Then he knows (we would say that this is the definition of division, but such concepts were not Dardi's) that

$$\frac{5}{3-\sqrt{4}}=3+\sqrt{4}$$

and that

$$\frac{5}{3+\sqrt{4}}=3-\sqrt{4}.$$

What we need to find is

$$\frac{8}{3+\sqrt{4}}.$$

So far, nothing amazing. But now comes something unexpected. Dardi makes appeal to the rule of three, which tells him that

$$\frac{8}{3+\sqrt{4}}=\left(8\cdot\left[3-\sqrt{4}\right]\right)\div 5=\left(24-\sqrt{256}\right)\div 5$$

which he then in agreement with *abbacus* algebra[3] aesthetics reduces to

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not say much. However, this manuscript uses the characteristic Venetian spelling *çenso* and the corresponding abbreviation *ç*. So does the Arizona manuscript, whose orthography is also northern; the last two manuscripts, written in Tuscan, still use the abbreviation *ç* even though writing *censo* or *cienso* when not abbreviating (actually I have not seen the Florence manuscript, but Libri's transcription of a short extract (1838, p. III, 349–359) uses "*c*," probably standing for *ç*). There is thus no reasonable doubt that the original was written in Venetian or a related dialect.

The manuscripts are discussed in Hughes (1987) and in Franci (2001, p. 3–6). I thank Van Egmond for access to his personal transcription of the Arizona manuscript.

[2] I refer to the text edition in Franci (2001, p. 59); the Chigi manuscript (fol. 12ᵛ, original foliation; probably closer to Dardi's own text) has $\widehat{m}$ instead of *meno* and *e* ("and") instead of *più* but is otherwise no different.

[3] "Abbacus" (*abbaco*, *abbacho*) has nothing to do with any variant of the reckoning board. It stands for practical arithmetic, but in the variant that was taught in the "abbacus school", and it calculated with Hindu-Arabic numerals on paper. Abbacus schools, existing between Genova-Milan-Venice to the north and Umbria to the south from ca 1260 to c. 1600, were frequented by artisans' and merchants' sons (also sons of patrician-merchants like the Florentine Medici) for 2 years or less around the age of 11–12.

$$4\frac{4}{5}-\sqrt{10\frac{6}{25}}.$$

What precisely was the rule of three for Dardi? Not the *problem* to find an unknown q (or p) from "if q is to p as Q to P" (where p and q may stand for "quantity" and "price", respectively), nor for *whatever method* can be used to solve that problem. The rule of three is the specific method which first multiplies and then divides, and only that. In the Italian *abbacus* school environment, it was taught in words like these:

> If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind), and divide in the other.

This is the formulation in the Umbrian *Livero de l'abbecho* (Arrighi 1989: 9),[4] dating from around 1300; it is repeated more or less verbatim in almost all *abbacus* writings that formulate the rule—see (Høyrup 2012, p. 148–152). This is thus certainly what Dardi referred to. The rule was taught unexplained; it is indeed difficult to explain, since the intermediate product has no concrete meaning.[5]

The recourse to the rule of three was certainly meant by Dardi as an explanation. Is it a demonstration? Probably even Dardi did not think of it in terms like that, but rather as what we might express as a "reasoned procedure".

We may compare with the way we ourselves may have been taught to perform the same division—I myself around the age of 14. We would have been told to multiply the numerator and the denominator of $\frac{8}{3+\sqrt{4}}$ by $3-\sqrt{4}$,

$$\frac{8}{3+\sqrt{4}}=\frac{8\cdot\left(3-\sqrt{4}\right)}{\left(3+\sqrt{4}\right)\cdot\left(3-\sqrt{4}\right)}=\frac{8\cdot\left(3-\sqrt{4}\right)}{3^2-\sqrt{4}^2}=\frac{24-8\sqrt{4}}{9-4}=\frac{24-8\sqrt{4}}{5}.$$

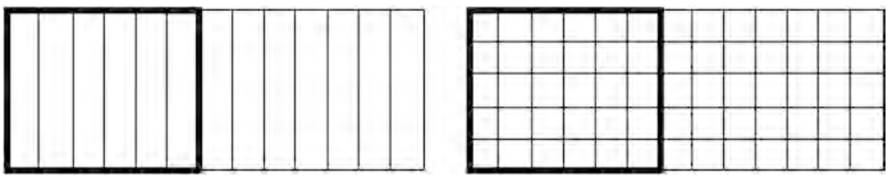
Even this is a reasoned procedure, but we might spontaneously tend to see it as more akin to demonstration. But how did we know that a fraction does not change its value when numerator and denominator are multiplied by the same number? And would $3-\sqrt{4}$ be a number in the sense corresponding to the argument behind this manipulation?

Abbacus *algebra* was not taught here, but flourished from ca 1310 onward in the environment of abbacus school teachers, serving to display their competence in the competition for pupils or for municipal employment.

[4] My translation, as other translations in the following unless otherwise stated.

[5] In contrast, the two alternative methods where division precedes multiplication can be explained meaningfully: q must cost p/P as much as Q; and Q/P is as much as can be bought for one monetary unit.

It certainly was not. At an earlier moment we may have been presented with an explanation of the expansion of, say, $\frac{6}{13}$ into $\frac{5\cdot 6}{5\cdot 13}$ corresponding to this diagram:



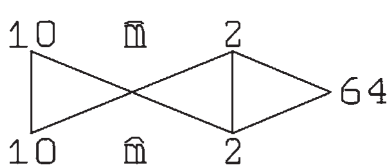
To the left, in heavy outline, we see $\frac{6}{13}$ of a rectangle—6 out of 13 equal strips. To the right the same, now 5·6 out of 5·13 equal squares, that is, $\frac{5\cdot 6}{5\cdot 13}$ of the rectangle.

To make that a rigorously valid argument in the case of irrational factors would require something like an Archimedean exhaustion. In any case, when we were confronted with $\frac{8}{3+\sqrt{4}}$ we had long forgotten the argument for the possibility of reduction or expansion of fractions (*if* we had ever been presented with one); we had just got accustomed—just as Dardi's model reader was accustomed to the use of the rule of three.

There is a difference, however, and that difference is elucidated by another passage from Dardi. Here, Dardi wants to "*prove* by a numerical example" that "minus times minus makes plus"[6]:

> I shall say, 8 times 8 makes 64, and this 8 is two less than 10, and multiply it by another 8, which is still 2 less than 10, which similarly shall make 64. This is the proof, multiply 10 times 10, it makes 100, and 10 times 2 less, it makes 20 less, and the other 10 times 2 less, it still makes 20 less, and you have 40 less, which 40 less detract from 100, remains 60. And to finish the multiplication, multiply 2 less times 2 less, which makes 4 more [*più*], join it above 60, and you have 64. And if 2 less times 2 less made 4 less, one should detract ⟨it⟩ from 60, and 56 would remain. Then it would seem that 10 less 2 times 10 less 2 would make 56, which is not true. And so it would be if 2 less times 2 less made nothing, then the multiplication of 10 less 2 times 10 less 2 would come to make 60, which is still false. So less times less by necessity needs to make plus [*più*].

This is followed in the Chigi and the Arizona manuscripts by a diagram.



[6] Franci (2001, p. 44). The words are "dimostrare per numero" and "meno via meno fa più"—in the Chigi manuscript (fol. 5[v]) "demostrar per numero" and "men via men fa più". Dardi distinguishes between *mostrar*, "to show", and *demostrar*, "to prove".

One may wonder at the stumbling logic in the final part of the argument—why not just *derive* that "less 2 times less 2" *must* make the 4 that has to be added to 60 in order to produce 64?[7] The other objection that might be raised—that a numerical example cannot be a proof—is easily discarded: the numerical values are just as peripheral as the actual lengths of lines entering a Euclidean proof. As Aristotle points out in *Metaphysics* M, 1078ᵃ17–21 (Ross in (Aristotle, *Works*, VIII)),

> if we suppose attributes separated from their fellow-attributes and make any inquiry concerning them as such, we shall not for this reason be in error, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the premises.

As long as the argument does not depend on the actual numerical values but these just serve to carry its structure, a proof "per numero" is as good or as bad as any Euclidean demonstration by diagram.

Let us therefore concentrate on the structure. One might argue (from the meaning of multiplication as repeated addition) that adding 10 2 times less amounts to adding 20 less, and that adding 2 10 times less also amounts to adding 20 less. However, Dardi offers no argument, and in the preceding section (where number less root is multiplied by number less root, with the example $\left(3-\sqrt{5}\right)\cdot\left(4-\sqrt{7}\right)$ one can see that the explanation (Franci 2001, p. 43) is merely

> You shall at first multiply the numbers one by the other, that is, 3 times 4, which makes 12, and save it. And then multiply in cross the numbers times the roots, which is less, and what results is root less. Therefore multiply 3 times less root of 7, which makes root of 63, […].

It is in the sequel that the need to multiply less by less arises. In contrast, less times more and more times less are treated as in need of no argument. They are familiar matter, just like the rule of three.

So, be it in reasoned procedure, be it in demonstration, the explanation makes use of what the learner (the presupposed or model learner) can be assumed to accept as evident—not necessarily because of preceding argument, the intuitively obvious may do as well; that is what I shall call the *locally obvious*. And, this is the crux of what precedes: *habit creates intuition* (though certainly not in one-to-one correspondence). The advice attributed to d'Alembert, "Allez en avant, et la foi vous viendra", is not too far away. Habits, on the other hand, are often linked to a *particular* practice, and thereby to the particular institutions that wield this practice. Dardi's use of the rule of three is an example, visible to us because we do not participate in abbacus school practice. Locally, it was obvious; at our distance, something that itself needs argument.

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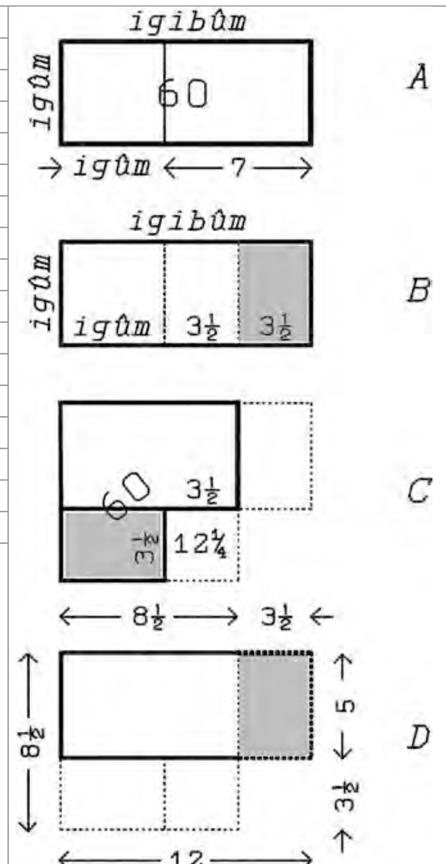
[7] Luca Pacioli, in (1494, p. I 113ʳ) actually does no better—he adds yet another possibility to be ruled out, namely that $(-2)\cdot(-2) = -2$ (Pacioli operates with negative, not just subtractive numbers), and is even more loquacious here than he usually is (to the point of being obscure).

2 Critique

What is obvious for one person (for instance, the teacher) may not be obvious to another one (for instance, the student); and what at first seems obvious may even become doubtful for the same person at second thoughts. That is where *critique* sets in, reflections about *Möglichkeit und Grenzen*, in Kant's words from the opening of the Third Critique (ed. Vorländer 1922, p. 1). I shall illustrate this with an Old Babylonian example[8]—the text YBC 6967, from somewhere between 1750 BCE and 1600 BCE. I quote the translation from Høyrup (2017, p. 45*f*).

Obv.
1. The *igibûm* over the *igûm*, 7 it goes beyond
2. *igûm* and *igibûm* what?
3. You, 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30′;
6. 3°30′ together with 3°30′
7. make hold: 12°15′.
8. To 12°15′ which comes up for you
9. 1` the surface join: 1`12°15′.
10. The equal of 1`12°15′ what? 8°30′.
11. 8°30′ and 8°30′, its counterpart, lay down.
Rev.
1. 3°30′, the made-hold,
2. from one tear out,
3. to one join
4. The first is 12, the second is 5
2.5. 12 is the *igibûm*, 5 is the *igûm*



This asks for explanation—that is the price of *Verfremdung*. On the tablet, numbers are written in a floating-point place-value system with base 60. In the translation,

[8] The "Old Babylonian period" is the period 2000–1600 BCE (according to the "middle chronology"); the mathematical texts come from its second half.

they have been provided with an absolute order of magnitude. In this translation, ´ indicates decreasing and ` increasing order of magnitude; "1`12°15´" thus stands for $1{\cdot}60^1 + 12{\cdot}60^0 + 15{\cdot}60^{-1}$; when not needed for separation or clarity, "°" is omitted ($60^0 = 1$); "12" is thus the same as "12°".

The problem deals with two numbers belonging together in the table of reciprocals—*igûm* and *igibûm*, meaning "the reciprocal" and "its reciprocal". We should expect their product to be 1, but it is actually meant to be 1` = 60 (as mentioned, the system was floating-point). Moreover, the *igibûm* exceeds the *igûm* by 7. The words ("to make hold", "surface", "counterpart"—see imminently) show the procedure to be geometric—the two numbers are represented by the sides of a rectangle with area 1` = 60. That is shown in **A**. In **B**, the excess 7 of the *igibûm* over the *igûm* is "broken", that is, bisected—not only the segment representing it but also the appurtenant part of the rectangle, resulting in two rectangles with one side equal to the *igûm* and the other to 3°30′ = $3^1/_2$. In **C**, the outer of these rectangles is moved—the two segments of 3°30′ are "made hold", that is, arranged so that they contain a rectangle (here a square) of area 3°30´ × 3°30´ = 12°15´. To this square is joined the original rectangle transformed into a gnomon; the result is a square with area 1` + 12°15´ = 1`12°15´. Then the "equal" of this larger area is found, that is, one of the two equal sides that contain it. It is 8°30´ = $8^1/_2$. This is "laid down" together with its "counterpart"—the term may mean "to draw" or "to write", possibly also to lay down on a reckoning board (in the actual case two boards, one for each). However that may be, in **D** the "made-hold", that is, the part that was moved, is put back into its original place. Removing 3°30′ from 8°30′ leaves 5, which is the *igûm*. Putting it back yields 12, the *igûm*. We may describe the whole procedure as "cut-and-paste geometry in a square grid".

On the surface, everything here is just "seen" to be correct—but since the drawing is not found on the tablet but either just imagined by "mental geometry" or sketched on a dust board or in sand strewn on the brick-laid courtyard, even this is an instance of the locally obvious, made *obvious for us* by being transferred to our familiar medium of drawings in true proportions.

Yet one thing hides below the surface. Normally, the Babylonian reckoners, as we, would mention addition before subtraction. This is also reflected in the reversal of the order in the last two lines, 12 resulting from addition, 5 from subtraction. But in lines rev. 1–3, subtraction is performed first. The reason is regard for concrete meaningfulness: we cannot put something back in place before it is made available.

This is not evidence of "a primitive mind not yet prepared for abstraction", as has been supposed. In analogous situations, earlier texts (from around 1775 BCE) simply say "to one join, from the other tear out" (as still reflected in the order to the last two lines). At some moment, some teacher, perhaps challenged by a student, perhaps as a result of his own second thoughts, has discovered that the inherited way of speaking is deprived of concrete meaning; that is, he has engaged in *critique*.

Critique is not a conspicuous characteristic of Old Babylonian mathematical texts. I know of one other instance. Some early problems add sides of squares or rectangles to their areas without qualms, and then proceed like here, treating the segments in question as "broad lines" provided with an inherent breadth of one length unit that can be bisected.[9] Even here, later texts change their way, providing explicitly the segments with a width equal to one. Since this is done in three different ways, it seems that no less than three different teachers, each belonging to a school tradition of his own, have engaged independently in critique.

But this is all I have noticed in the Old Babylonian mathematical corpus as far as indubitable critique is concerned. After all, mathematics was basically taught as a means for administration, and even though it created a higher level of "supra-utilitarian" problems, the norms governing the practice out of which these grew asked for finding "the right number", not for theoretical justification beyond what might be didactically useful.

3 Demonstration, Critique, and the Culture of Liberal Arts

From Classical Antiquity we have the concept and ideal of "liberal arts", knowledge bodies that have no technical use but are considered goals in themselves. We may leave aside what later times did to the concept, from the Latin Middle Ages to our own world, and stick to the ancient ideal and its reality.

We should take note that the famous "cycle" of seven Liberal Arts (grammar, rhetoric, dialectic; arithmetic, geometry, astronomy, harmonics) was only formed during or after Plato's mature years, and that the supposed "seven" were normally two and nothing more—grammar, that is, good and correct use of language, and rhetoric. Augustine was no exception when he had to study on his own everything beyond these subjects (*Confessions* IV.xvi, Rouse 1912, p. 198)—but he certainly was when complaining about it. Nor was he an exception when, though an intellectually ambitious teacher of the Liberal Arts, he never had students interested in anything going beyond these matters. Things have to be reduced to due proportions.

Yet there *were*, as we know, people engaged in "liberal" mathematics during Classical Antiquity—and not only Euclid, Archimedes and Apollonios. According to Reviel Netz's estimate (1999, p. 282*f*) of the number of those who at some moment in life made a piece of explicitly reasoned mathematics, 144 have left at least minimal direct or indirect traces; perhaps some 300 were still known by name in Late Antiquity—and in total perhaps 1000, one born on the average per year, but certainly with a more uneven distribution than simple randomness would suggest (and quite possibly considerably fewer).

Their appearance also precedes the formation of the *cycle* of Liberal Arts. It *almost* had to, how could the quadrivial arts (arithmetic, geometry, astronomy and harmonics) become part of the cycle if they did not already exist? Yet we should beware that what entered the cycle were, at least by name, Pythagorean fields of interest, and to which extent these corresponded to the reasoned theoretical fields we know from Aristotle's time onward can be disputed. Even the nature of the mathematics which

[9] This notion of "broad lines" and its rather widespread occurrence is discussed in Høyrup (1995).

according to Plato should be taught to the guardians of his republic (*Republic* VII, 525D5–E3) is subject to doubt—cf. (Mendell 2008). There is no compelling evidence (if we do not count as such much later Neoplatonic interpretations) that his "arithmetic" was something like the theory of *Elements* VII–IX—after all, the word basically means "counting", and how far this meaning was stretched by Plato is not clear from his text. To Henry Mendell's arguments we may add a passage from Aristotle's *Metaphysics* N, 1090^{b}27–29 (which no longer concerns the state of affairs at the moment when the *Republic* dialogue is supposed to have taken place but Plato's own teaching at the moment when Aristotle was working at the Academy or later). After other objections against Plato's identification of numbers with ideas it is pointed out (trans. Ross in [Aristotle, *Works*, VIII]) that

> not even is any theorem true of them, unless we want to change the objects of mathematics and invent doctrines of our own.

That is: whatever Plato maintains in his mature philosophy about number has nothing to do with the theoretical arithmetic that had been created no later than the fourth century BCE.[10]

In any case we know that some kind of theoretical mathematics existed during the second half of the fifth century BCE. Famously, the possibility of *incommensurability* had been discovered by then, most likely by Pythagorean *mathematikoi*, and we know that first Theodoros and later Theaitetos worked on this topic—Plato's dialogue *Theaetetus*, though written after 370 BCE, can be considered testimony. From the reports about and fragments from Archytas (Diels 1951, p. I, 429–438), we also know about investigation of the three main mathematical *means* (arithmetic, geometric, harmonic). At least irrationality is beyond what could be of interest in any productive or administrative practice; a connection between the theory of means and the theory of harmonics can be presupposed, but then the theory of harmonics was a mathematical theory, and its relation to practised music questionable (questioned indeed by Aristoxenos). The theory of means was also linked to the search for two mean proportionals, which Archytas treated; even this was of no interest for administrators or master builders.

We are ignorant, however, not only of the precise arguments used by Theodoros and Archytas but also of their overall argumentative style. As regards Meton and Euctemon, we are even worse off concerning the kind of mathematical argument (if any) they used together with their astronomical observations; in any case we cannot

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[10] A number of attempts have been made to save Plato by proving that Aristotle does not understand him—see, for instance, Tarán (1978) with references to others sharing his view, or the list in Cherniss (1944, p. X). Such attempts are misguided, what Aristotle does is to point out that the numbers Plato speaks about have nothing to do with what others mean by number—no more, indeed, than the "self-moving number" which the Pythagoreans identify with the soul (*De anima* 408^{b}32*f*).

A different question, which however does not concern us here, is whether the traces we have of Plato's views can be given a coherent and historically possible interpretation. Beyond the discussion and references in Tarán and Cherniss, see Mendell (2008, p. 128 n. 3).

say that their work belonged under the heading *theory*, the calendar was certainly a practical concern.

Happily, we know more about Hippocrates of Chios. We have his investigation of the lunules as rendered via Eudemos by Simplicios (Thomas 1939, p. I, 239–253). It is obviously reasoned—the three "classical problems", one of which (the squaring of the circle) is the inspiring background to Hippocrates's question, only make sense as theoretical problems. But there is no trace of axiomatics, the argument makes use of two principal tools, together with some properties of his diagrams which he tacitly takes for granted as intuitively obvious. One tool is that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the legs of the right angle—the "Pythagorean theorem"; the other is that the area of a circle is proportional to the square on the diameter.[11] Both had been staple knowledge for Near Eastern surveyor scribes at least since the Old Babylonian period—both are indeed used in mathematical problems from that epoch, and the proportionality of the areas of similar figures to the square of a characteristic linear dimension (side of a square, perimeter or diameter of a circle) is the fundament for the geometric part of the tables of technical constants. So, Hippocrates may have made use (*systematic* use, which is where he differs from for example Dardi) of the locally obvious); to believe that he must have known or produced a proof, for instance for the proportionality of the circular area to the square on the diameter is a *petitio principii*, proving that Greek geometry already had the character we know from the third century BCE from the tacit assumption that it had.

Further, we have Eudemos's ascription to Hippocrates of a first collection of elements—an ascription we know from Proclos's *Commentary on Book I of the Elements* 66 (Morrow 1970, p. 54). This collection is likely to have been connected to Hippocrates's teaching in Athens. The direct evidence for such teaching is a reference in Aristotle's *Meteorology* to "those around Hippocrates and his disciple Aischylos".[12] The members of this circle cannot have been engaged in practical mathematics: firstly, then they would have had no need for a collection of elements: secondly, Aristotle speaks about their opinions concerning comets. So, this earliest almost direct reference to teaching of geometry also shows it to have been teaching of geometry as a "liberal" subject.

[11] Thomas Heath (1921, p. I, 201) argues from Hippocrates's text that he knew what was to become propositions III.20–22, 26–29 and 31 in Euclid's *Elements*. This would not be amazing, they can be derived from the equality of the angles at the basis of an isosceles triangle by means of the same kind of counting as Hippocrates wields when applying the Pythagorean theorem. But it is equally possible—not least because Hippocrates makes use of these properties of figures without noticing that an argument might be needed—that he made use of what could "be seen" without having recourse to formulated propositions.

[12] Bekker (1831, p. 342^b36–343^a1). "Those around" was the standard way to refer to the circle of those who studied with a philosopher or similar teacher. Strangely, the Loeb as well as the Ross translation omits "those around", even though the Loeb edition conserves it in its Greek text. The secondary literature on the other hand (including myself on earlier occasions) has spoken about Hippocrates's teaching without questioning it.

We have no direct evidence concerning the possible teaching of Oinopides, also from Chios and slightly older than Hippocrates—at most the suggestion of Paul Tannery (1887, p. 109) that Hippocrates learned from him. Relying on Eudemos, however, Theon of Smyrna (Dupuis 1892, p. 320*f*) states that Oinopides discovered the obliquity of the ecliptic. That the planets do not move on the celestial equator was too obvious to be a discovery, so two interpretations of this passage are possible: Oinopides may have discovered that the motions of the planets not only run through a specific sequence of celestial signs (that is how matters were seen by Babylonian mathematical astronomers) but describe a great circle (which the Babylonians could not think, not possessing the notion of the heavenly vault as a sphere or hemisphere); or he may have measured the obliquity of the ecliptic (which is however so easy to do once the idea of an oblique great circle is conceived that it can hardly count as an independent discovery[13]). Our present point is a scene depicted in Plato's *Erastae* (Lamb 1927, p. 312*f*), set in the later fifth century BCE. It portrays two boys in "the grammar school of the teacher Dionysios" eagerly discussing an astronomical problem "either about Anaxagoras or about Oinopides" involving the obliquity of the ecliptic. This school (also Plato's own school according to Diogenes Laërtios (Hicks 1925, p. I, 278*f*)) was a school for "the young men who are accounted the most comely in form and of distinguished family" (thus *Erastae*), not one teaching banausic trades; here, things like Oinopides's astronomy were thus taught at least at a level that allowed eager discussion.

A different kind of evidence comes from Aristotle's writings.[14] The ideal organization of a field of knowledge as prescribed in the *Posterior Analytic* is obviously inspired by geometry[15]—not just reasoned geometry but axiomatic geometry. During the century or so that had passed since Hippocrates wrote his elements, many things could of course have changed, and Aristotle presents much material elucidating the process.

Quite a few of Euclid's definitions (or alternatives referred to by commentators) were known to Aristotle. I shall mention only two examples. Firstly, *Topica* $143^{b}11$*f* refers to those who define the line as a "length without breadth", μήκος ἀπλατές, exactly Euclid's definition I.1.[16] Secondly, though paraphrased and contracted, the definition of geometrical similarity referred to in *Analytica posteriora* $99^{a}13$*f* is obviously the one offered in *Elements* VI.[17]

Definitions had been a concern in Greek philosophy for quite some time. According to Aristotle's *Metaphysics* $987^{b}3$, (trans. Ross in [Aristotle, *Works*, VIII]), "Socrates […] fixed thought for the first time on definitions". Whether he

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[13] All that is needed is to measure the culmination of the sun at summer and winter solstice and to halve the difference.

[14] From Plato's dialogues, too. But they are often (already, and perhaps mainly, because of the half-poetic genre) too ambiguous to be of much use in the present discussion.

[15] "Inspired", not copying, already for the reason that Aristotelian syllogistic logic does not fit the way geometric proofs are argued. But also for other reasons, cf. McKirahan 1992, p. 135–143.

[16] Respectively Bekker 1831, p. I, 143 and Heiberg 1883, p. I, 2.

[17] Respectively Bekker 1831, p. I, 99 and Heiberg 1883, p. II, 72.

was really the first or inspired by contemporary mathematicians is probably not to be decided—not least because Aristotle speaks of όρισμοί but Euclid (and plausibly geometers before him) of Ὅροι, which rather means "delimitations". Aristotle is likely to have been aware that the difference was more than just a choice between synonyms.

Among Euclid's common notions, the third ("if equals be subtracted from equals, the remainders are equal" (Heath 1926, p. I, 223) is Aristotle's paradigm for an axiom or "peculiar truth" valid within a particular genus. It serves as such in *Analytica posteriora* 76^{a}41, and again in 76^{b}20*f*, but also in *Analytica priora* 41^{b}21*f* as an example of a presupposition that has to be made explicit in order to avoid a *petitio principii*.

Further, Aristotle knew Euclid's second postulate—that can be seen in *Physica* 207^{b}29–31 (Hardie & Gaye in [Aristotle, *Works*, II]):

> [mathematicians] do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.

Euclid (Heath 1926, p. I, 154) requests (that is the meaning of "postulate") that it be possible "to produce a finite straight line continuously in a straight line".

As far as I know, the other postulates are not quoted (nor paraphrased) in the Aristotelian corpus; one, moreover, is absent where it would have been adequate to mention it, namely in *Analytica priora* 65^{a}4–7 (Jenkinson in [Aristotle, *Works*, I]). This passage refers, as an example of hidden circular reasoning, to

> those persons […] who suppose that they are constructing parallel straight lines: for they fail to see that they are assuming facts which it is impossible to demonstrate unless the parallels exist.

Postulate 5 (Heath 1926, p. I, 155),

> if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles

was obviously meant to repair that calamity. Actually, it only does so halfway. It excludes hyperbolic but not elliptic geometries (precisely those where parallels do not exist). For this purpose, one has to presuppose, for example, that two straight lines cannot enclose a space, which some geometers indeed added as an axiom according to Proclos, *Commentary on Book I of the Elements* 183 (Morrow 1970, p. 143), and which in fact is used in a dubious passage in the proof of *Elements* I.4 (Heath 1926, p. I, 248, cf. p. 249) (apparently a scholion that has crept into the text).

What can we derive from these observations? In general that geometry as known to Aristotle was already striving for axiomatization—no wonder, we know from Eudemos as quoted by Proclos (*Commentary* 67, [Morrow 1970, p. 56]) that at least Theudios made a new, better arranged collection of elements, and that a number of mathematicians worked together at the Academy in Plato's time. But we also see that the enterprise had not yet led to the goal, at least not as a social undertaking—those who undertook to construct parallel straight lines while presupposing unconsciously that such lines exist were still building their reasoning on the locally obvious—and so was even Euclid in many cases, for example when he took it for granted that two lines cannot enclose a space (not to speak of his many topological intuitions).

We may also have a look at Euclid's postulate 4 (Heath 1926, p. 155), "That all right angles are equal to each other". For us, this is locally obvious—"of course, they are all 90°". Apparently, it was just as obvious until the mid-fifth century BCE—and for a similar reason. Then, according to Proclos (*Commentary* 283, [Morrow 1970, p. 220*f*]),[18] Oinopides introduced the *construction* of the perpendicular by means of ruler and compass, calling the perpendicular a line drawn "gnomonwise"—implying that until then it had been made by means of a set square (γνώμων), in which case the equality seems obvious. However, with the new construction arose the need for a *definition* of what a right angle is. In Euclid we find this (*Elements* I, def. 10, Heath 1926, p. I, 153):

> When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

This seems to solve the problem, now we know what a right angle is, much better than the Old Babylonian surveyor-scribes (and probably the surveyor-scribes of the mid-first millennium), whose field plans show them to have distinguished between "wrong" and "right" angles, the former—those which are evidently skew—being irrelevant for area calculation and the latter—right for practical purposes—essential; see for example Høyrup (2002, p. 228). But it creates a new problem: Now it is no longer obvious that all right angles are equal, and that is needed in many proofs.

The preceding three paragraphs lapsed into old-style historiography of mathematics, which tended to forget that mathematical knowledge and practice do not exist per se but have social carriers—or, if mentioning persons, would take it for granted that these, as "mathematicians", would think "like mathematicians". The reader who had no objections will recognize how easily this lapse occurs.

Yet a problem is only one if it is a problem *for somebody*, and it only becomes a problem in the encounter with that somebody. Here, we may return to the boys from *Erastae*. If they could discuss eagerly about Oinopides and his work on the obliquity of the ecliptic, they might also challenge their teacher, and ask (this was shortly after Oinopides introduced his construction) *what* this right angle is *in itself* which he constructs (apart from being supposedly useful in astronomy, as Proclos says Oinopides had thought). The answer would be something like the Euclidean definition. And then, at a later moment, similar eager students might discover that with this definition, the equality of right angles is no longer obvious. This is *critique*, born as an endeavour from the character of the environment.

We may further remember that the environment of philosophers (to which we may count the theory-oriented mathematicians teaching elite youth just as did other philosophers) did not strive for truth in peaceful collaboration but in competition and strife. Here, *critique* would coincide with *criticism* or *challenge* of colleague-competitors.

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[18] Cf. also von Fritz 1937, p. 2265 *f*.

4 Axiomatization

Critique had been a driving force in the axiomatization of geometry—axiomatization *as a goal* had not been imaginable when Oinopides and Hippocrates made their work. Not only was axiomatization the outcome of a process yet in their future; so was the discovery of the *idea* of axiomatization as a possibility. Plato's reproach to geometricians in the *Republic* (533C–D, trans. (Shorey 1930, p. II, 203), that they are

> dreaming about being, but the clear waking vision of it is impossible for them as long as they leave the assumptions which they employ undisturbed and cannot give any account of them. For where the starting-point is something that the reasoner does not know, and the conclusion and all that intervenes is a tissue of things not really known, what possibility is there that assent in such cases can ever be converted into true knowledge or science?

—this reproach may look as if Plato had observed the strivings of contemporary mathematicians to achieve axiomatic order (even though the "assumptions"/ Øπόθεσις he speaks about may also be local, as in Hippocrates's text). Whether an axiomatic structure or just locally coherent argument is meant, Plato does not accept such geometry as more than a mere mental exercise preparing the best souls for the study of dialectics,

> the only process of inquiry that advances in this manner, doing away with hypotheses, up to the first principle itself in order to find confirmation there,

which first principle is insight into "the good", no formulated axiom; and dialectic as imagined by Plato is in consequence no axiomatic system.[19]

Aristotle understood that this was a pipe dream, and that explicit axiomatization is the maximum that can be achieved. This is indeed pointed out in the very first sentences of his *Analytica posteriora* ($71^{a}1$–9, Mure in [Aristotle, *Works*, I]):

> All instruction given or received by way of argument proceeds from pre-existent knowledge. This becomes evident upon a survey of all the species of such instruction. The mathematical sciences and all other speculative disciplines are acquired in this way, and so are the two forms of dialectical reasoning, syllogistic and inductive; for each of these latter makes use of old knowledge to impart new, the syllogism assuming an audience that accepts its premisses, induction exhibiting the universal as implicit in the clearly known particular.

As we notice, this is in itself an instance of inductive dialectic as here explained.

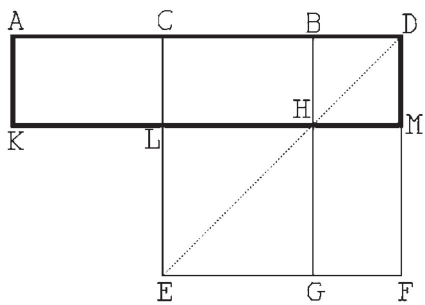
In spite of the ambiguity of Plato's polemics (which we need not reproach him, his discourse has other concerns), these words together with the rest of the *Analytica posteriora* leave no doubt that in the mid-fourth century BCE not only Aristotle but also the geometers were familiar with the axiomatic ideal. From now on, it provided a possible format when new fields were taken up and did not need to be the unplanned outcome of a process driven by other forces. As we know, this format was to be used for example by Archimedes.

[19] One can argue from certain Platonic texts—but this would lead us astray—that this insight in "the good" is achieved via mystical experience. As a hint, observe the force of the images of *light*.

Critique, as argued above, had been a motive force in the process ending up in axiomatization before this process could be driven by a recognized aim. But critique was more than that. A look at *Elements* II.6 (Heath 1926, p. I, 385) will illustrate it:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

Whoever encounters these lines for the first time is likely to ask why this seemingly abstruse theorem is interesting. However, if we look at the diagram that accompanies the proof we recognize a familiar situation (I follow Heath, but emphasize some lines and weaken another one for clarity of the argument). Here, the bisected line is represented by *AB* and the added line by *BD*. *DM*, perpendicular to *AD*, equals *BD*. *AB* is thus the excess of length over width in the rectangle *ADMK*. If we identify *AD* with the *igibûm* and *DM* with the *igûm*, we are back at the Old Babylonian problem discussed above, and *AB* must be 7.



There are differences, however. Firstly, Euclid does not solve a problem: if we impress algebraic categories on his text, then he presents us with an identity. This identity can of course be used to solve problems by taking some of the magnitudes involved to be known and others unknown (for example, taking *AB* to be 7 and the area *ADMK* to be 60 will allow us to find *AD* and *BD*).

Secondly, Euclid does not move segments or areas around. At first he *constructs* the square *CDFE* on *CD*, which ensures that the angle *ADM* is really a right angle. He then draws the diagonal *DE*, which has no place in the Old Babylonian procedure. He then draws the line *BG* parallel to *CE* or *DF;* through the intersection *H* of *BG* and *DE* he draws *KM* parallel to *AB* or *EF*, and through *A* the line *AK* parallel to *CL* or *DM*. That allows Euclid to show that rectangle *ACLK* is equal to the rectangle *HMFG*, and thus that the rectangle *ADMK* equals the gnomon *CDFGHL*, whence finally the equality claimed in the enunciation. Nothing is cut, moved around and pasted, all is proved to the best standards of theoretical geometry as these had been shaped in the late fifth and the fourth centuries BCE. The proposition thus functions as a critique of the cut-and-paste procedure by which the problem was traditionally solved, showing why and under which precisely stated conditions it works—thus saving it instead of rejecting it as Plato did when he reproached geometers their "talk of squaring and applying and adding and the like" (*Republic* 527B, (Shorey 1930, p. II, 171)).

That it was also *meant* as critique and saving appears to follow from analysis of the whole sequence *Elements* II.1–10. A discussion in depth would lead too far, but see Høyrup (2002, p. 400–402). A blunt summary goes like this:

– All 10 propositions correspond in the way just sketched for II.6 to riddles or basic cut-and-paste-tricks belonging at least since ca 1800 BCE to an environment of surveyors—riddles which once inspired the Old Babylonian scribe school, but have also left their traces in a variety of written mathematical cultures until the Late Middle Ages, including Greek pseudo-Heronic practical geometry (and were therefore certainly known to Greek theoretical geometers);
– propositions 4–7 are used later in the *Elements*, mainly in Book X, the others not[20]; like many of the definitions of Book I that are never used afterwards, they represent something familiar that has to be saved for its own sake;
– propositions 2 and 3 are special cases of proposition 1; propositions 4 and 7 are different formulations of what is practically the same matter; the same can be said about propositions 5 and 6 and about propositions 9 and 10. None the less, all are proved independently, as if not only the results but also the traditional methods had to be saved through critique.

So, between Aristotle's and Euclid's times, deductivity completed as axiomatization established itself as the norm for how mathematics should be made—obviously only within the tiny group which we, like Netz, would normally accept as "mathematicians". Most of those who went through the normal syllabus of Liberal Arts would not care about anything beyond rhetoric, as pointed out above—and within that minority which had greater ambitions, most would stop at knowing a few concepts and enunciations and not care for demonstrating. That is clear from the relative popularity of Nicomachos's writings, from handbooks like those of Martianus Capella and Cassiodorus, and from Theon of Smyrna's explanation of the mathematics needed for the study of Plato. Among those who calculated or constructed for administrative or productive purposes, the norm never took root, at most we find arguments from the locally obvious—to see this, we may look at Vitruvius and the pseudo-Heronic writings.

In Euclid's time already, the effects of the "liberal" curiosity of the fifth century BCE had subsided and been replaced by institutionalized norms. For that reason, the importance of critique as a partner and root of axiomatization seems also to have subsided (after all, the critique in *Elements* II is almost certainly borrowed from late fifth or early fourth-century predecessors, as the proportion theory in *Elements* V is supposed to be borrowed from Eudoxos). Heron's *Metrica* may to some extent be considered a rewriting of practical geometry *vom höheren Standpunkt aus*—but only to a quite limited extent in a way that allows us to speak of critique.

[20] Cf. Mueller 1981, p. 301.

5 And Then?

Not too long after Euclid's third century BCE, Greek mathematics entered the age of commentaries or, in Reviel Netz's terms (1998), of "deuteronomic texts" (a somewhat broader category, encompassing also epitomes, etc.). In Simplicios's presentation of the Hippocratic fragment (early sixth century CE), he states (Thomas 1939, p. 237) that

> I shall set out what Eudemus wrote word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements* on account of the summary style of Eudemus, who set out his proofs in abridged form in conformity with the ancient practice.

That illustrates a partial change of norms. Commentaries fill out and explain; at times they also discuss. Even though Simplicios is engaged in a commentary to Aristotle, he follows the commentator habits and norms even here, but mainly by filling out and, implicitly, explaining. "Adding […] a few things from Euclid's *Elements*" means that Simplicios inserts the Hippocratic text in the axiomatized framework.

In its own way, the addition of commentaries and the standardized structuring of mathematical texts (Netz 1998, p. 268–270) is a new level of critique, arguing now *why and in which sense* the classical text that is commented upon is right and conformable to norms. But since this classical text has somewhat sacred status, this critique is uncritical—quite different from the critical critique of fellow-philosophers or teachers in the fifth to fourth centuries BCE.[21]

It is hardly necessary to point out that norms only govern practice to some extent; many causes—conflicting norms, incompetence, personal conflicting interest, and so forth—make actors deviate from them. Eudemos's lack of reference to the propositions which Simplicios feels he needs to insert may be due to fidelity to his source—he is writing a history of geometry and may have written more like a historian than as a mathematician. But it may also reflect that axiomatization in his time was still a developing *practice* and not yet fully effective as a norm. In Simplicios's age of deuteronomic texts, in contrast, the norm had become so explicit that we may see it as an *ideology*, an inextricable amalgam of the descriptive and the prescriptive, of "is" and "ought to". That ideology is still with us, admittedly more effective when governing the writing of textbooks (deuteronomic, indeed) than in mathematical research.

This ideology not only amalgamates the descriptive and the prescriptive levels. It also corresponds to the interpretation of ideology as "false consciousness". Most obviously, it disregards informatics, quantitatively the major part of twenty-first century mathematics. Already in 1970, a textbook from that field declared (quoted from the reprint (Acton 1990, p. xvii)) that

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[21] Genuinely critical stances had not disappeared—but they had become external, attacking the whole undertaking, not trying to save or to find the "possibility and limits" of mathematical knowledge. The best example is probably Sextus Empiricus (Bury 1933, p. IV, 244–321). This is harsh but informative and informed criticism—but not critique.

> It is a commonplace that numerical processes that are efficient usually cannot be proven to converge, while those amenable to proof are inefficient […]. The best demonstration of convergence is convergence itself.

This was written at a moment when the students using the book were supposed to work in FORTRAN, PL/1 or ALGOL—when programming was thus still transparent compared to what we find today. Every time your computer screen freezes, remember that the reason is probably an unpredicted conflict somewhere on the path from machine code through compiler to operating system or application—thus proof that the software has not been derived axiomatically from first principles. The role of beta-versions is to locate the conflicts ("bugs") that are most likely to occur—but this "critique through practice" never succeeds in doing more. The demonstrations of algorithm design remain local.

Even if we try to save the honour of mathematics by excluding informatics, the ideology misrepresents reality. In 1545, Cardano's *Ars magna* was printed. Then, gradually, the power of the tools offered by Descartes' *Géométrie* (1637) (also in analysis of the infinite and the infinitely small) was revealed. First, this transformed fundamentally what *algebra* could be; soon it also changed the global character of mathematics. Until the late nineteenth century, this whole process was founded (when not on controlled guess, as often happened) on arguments and demonstrations of no more than "local" validity, that is, premises that it seemed reasonable to accept or at least to try, but which were not built on clearly formulated first principles. Critique gradually improved the situation (even this was an epoch of competing scholars), but only the late nineteenth century was once again able to reshape mathematics on an axiomatic footing.[22]

In its merger of description and prescription, the ideology of thorough demonstration and demonstrability thus becomes false consciousness. The prescriptive aspect not only imposes a particular interpretation of the facts on the description—that probably cannot be avoided. It distorts it in a way that is easily looked through *if only one wants to*.

Recently in Italy, a nun when told by physicians that her supposed stomach ache were birth pangs, exclaimed "it is not possible, I am a nun!" Her false consciousness cannot have survived the next few hours. In general, false consciousness survives on Darwinian conditions: in some way it has to be useful. The one we have looked at here provides mathematics (that is, the mathematical establishment) with a comforting self-image, which can be projected (while the inconvenient baby, informatics, is given into adoption). It also serves to ostracize mathematical cultures that deviate from what the ideology prescribes and what we therefore claim *de*scribes

[22] These sweeping statements go beyond what can be documented in a few footnotes. But see Stedall (2010) for the development of algebra from Cardano to the early nineteenth century. Høyrup (2015, p. 29–33) covers an often overlooked aspect of the shaping and gradual reception of a Cartesian tool (the algebraic parenthesis). The painful advance in the foundation of infinitesimal calculus has been amply discussed; see, for example, Boyer (1949), Bottazzini (1986) and Spalt (2015)—not to speak of the innumerable publications dealing with particular aspects or figures.

our mathematics; thereby it serves a more direct and more indisputably political "projection of power".

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