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BABYLONIAN ALGEBRA FROM THE VIEW-POINT OF GEOMETRICAL HEURISTICS

AN INVESTIGATION OF TERMINOLOGY, METHODS, AND PATTERNS OF THOUGHT

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Dedicated to Professor Kurt Vogel on the occasion of his 95th birthday

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* A full literal retranslation of the text is given as a basis for the discussion.

INTRODUCTION

Traditionally, Babylonian algebra is believed to be an early instance of arithmetical algebra, and its use of "side", "length", "width" and "surface" as standard-names for variables is considered no more geometrical than the use of similar terms in Medieval rhetorical algebra, or the polygonal numbers of Greek arithmetic.

This essay, on the contrary, investigates the hypothesis that the basic terms of Babylonian algebra might have been meant literally, as geometrical designations, and not as metaphors for elements entering arithmetical relations; and that the methods of Babylonian algebra were rather based on a geometrical heuristic than on abstract arithmetical manipulations like those of later rhetorical and symbolical algebra.

The method of the essay consists primarily in a close reading of a number of Babylonian algebraical texts. For this purpose, most of these are retranslated from the published transliterations with close reference to the original vocabulary. Others are reported with reference to noteworthy points of method and formulation.

It turns out that the terminology of the Old Babylonian texts cannot be that of an arithmetical algebra, since it contains conceptual distinctions with no place inside such a framework; on the other hand, these conceptual distinctions agree very well with a geometrical interpretation of the procedure and pattern of thought. Many other details in the texts point in the same direction.

The fundamental procedures of the geometrical heuristic are investigated. They appear to include cut-and-paste-procedures (partition, completion and rearrangement of figures), considerations of scale or proportionality, and a technique of "accounting" analogous to our calculation of coefficients.

In some domains, though, concerned with problems and transformations of the first degree, real arithmetico-algebraic thought seems also to be manifest. Furthermore, the geometrical entities dealt with in the geometrical heuristics are used to represent quantities of other sorts - so, lines with a length are taken to represent prices or surfaces, just as in arithmetical algebra numbers or symbols representing abstract numbers may represent prices or geometrical quantities. Thus, Old Babylonian algebra really deserves the name of an algebra: Its geometrical basis constitutes an abstract means of representation.

Possible traces of a tendency towards an arithmetization of algebraic thought already in the Late Old Babylonian period are discussed, and in a Seleucid text a shift to purely arithmetical conceptualizations (but not necessarily away from geometrical heuristic as a method is revealed. In the end of the main paper, the implications of the whole investigation for the much-debated problem of the Greek "geometrical algebra" are pointed out.

As a spin-off, the interpretation of some of the mathematical texts from Susa in TMS is questioned. In particular, the supposed opposition between an "Akkadian" and a "Susian" method is shown to be without textual foundation, and a possible alternative interpretation of the term "Akkadian method" is proposed, viz. the method of quadratic complementation of a mixed second-degree equation.

In an appendix, the new interpretation of the algebraic terminology is used in an investigation of a selection of Old Babylonian real geometrical texts, and close connexions between the "real geometry" and the geometrical heuristic of the algebra is revealed.

In my translations, I have tried to keep one single term as the translation of each mathematical term of the original texts, and different translations of different terms (logographic equivalences apart when subject to no reasonable doubt). Furthermore, I have tried to render as faithfully as possible the metaphorical and connotative values of the original terms. This has the advantage to reflect in the translations the conceptual distinctions and the global conceptual structure of Old Babylonian mathematics, and to give the reader access to a non-technical side of the mathematical vocabulary which was possibly still alive in the Old Babylonian era (it will turn out that it was at least to some extent alive, and that the vocabulary was far from fully technicalized).

The strategy of my translations has, however, also a serious disadvantage - an obvious reason why all previous translations accepted far-ranging compromises in this respect. Of course, the metaphorical values of Babylonian mathematical terms will often differ from those of modern English. Worse, the conceptual distinctions of Old Babylonian algebra correspond to neither modern language nor modern mathematical thought; so, two different sorts of additions, a fluent variety of subtractive ideas and four different "multiplications" were kept apart by the Babylonian mathematical authors. In order to reflect this in an English vocabulary, I was led to the creation of a number of awkward terms, which are discussed along with their introduction. Until p. 30, the Akkadian or Sumerian terms corresponding to the translations are given parenthetically. Later, this is only done exceptionally, and the reader is referred to Table 1 (p. 107), which, for each standard translation used in the translations, gives the corresponding Akkadian and/or Sumerian term. Table 2 (p. 109) lists these terms with references to their occurrence in the translated texts as well as to passages where they are discussed. In some cases, a short supplementary discussion of the term is offered, and in a few a proposal for a revised standard translation (different from the one I choose in the early phases of the project) is given.

The basic version of the paper was written from July to September 1982. In June 1983 an emended version was prepared (by a method familiar from Old Babylonian algebra: cutting, pasting and adding) for a workshop on "Mathematical Concepts in Babylonian Mathematics", which took place in the Seminar für vorderasiatische Altertumskunde und altorientalische Philologie der Freien Universität Berlin, August 1 to August 5, 1983. A final, thorough revision, consisting however more in the addition of shorter or longer marginal commentaries than in re-writing or correction, was prepared in October 1983 to January 1984. Still, the appendix on Babylonian geometry, of which a much shorter version had been worked out for the workshop, was completely rewritten during the first phase of the final revision (and so, even the appendix contains a number of last-minute-marginal notes).

I am aware that the resulting form of the paper is far from elegant - clumsy might be an adequate description. Still, so much material is collected in one place that it could hardly be compressed into one journal publication. On the other hand, my investigation is not yet ripe for regular book-size publication. So, as a compromise, I have decided to circulate this final version as a discussion paper and as a background for further work in the field.

As it was stated in the first version, I am no Assyriologist. So, many many errors in Akkadian grammatical forms and interpretations may be present in the following text. For such errors, I ask for indulgence and correction (for bad English, I will only ask for the former, even though I shall wellcome the latter too). Any corrections, as well as other commentary, may be sent to

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Already during the preparation of the first version, I have benefited from conversations with Professor Olaf Schmidt, Lektor Mogens Trolle Larsen, Lektor Aage Westenholz and Dr. C.M. Taisbak, all of Copenhagen University, for which I am very grateful. As a response to the circulation of the preliminary version, I received a number of reactions, among which I will mention with special gratitude a number of further references from Professor Kurt Vogel and a very compact letter from Professor Wolfram von Soden, which was the occasion for many addenda as well as for several corrections and much further thought.

Special thanks are due to the participants in the Berlin workshop, especially Dr. Kilian Butz, Professor Jöran Friberg, Professor Wolfgang Lefèvre, Professor Hans Nissen, Professor Marvin A. Powell, Professor Johannes Renger and Dipl.-Phys. Jürgen Renn. First of all, Dr. Peter Damerow shall receive the expression of my gratitude, both for inspiring discussion and for the organization of the workshop, which grew to something much greater from an initial idea of a private working session.

Finally, I shall express my gratitude to Dr. Bendt Alster and, once more, to Lektor Mogens Trolle Larsen, who answered a number of final questions in the ultimate phase of the work.

Not least because nobody but the author read the final version of the manuscript in its entirety, the reader will easily guess who remains responsible for all errors.

The problem

It is an old observation that traditional algebraic problems can be solved by basically different methods. So, if we look at a problem of the type $x+y=a$, $xy=b$, we, of course, would solve it by manipulating symbols. Most Latin and Arabic Medieval mathematicians, from al-Khwârizmî onwards, would formulate it that "I have divided 10 into two parts, and multiplying one of these by the other, the result was 21" (Rosen 1831:41); in order to obtain the solution, they would call one of the numbers "a thing", the other "10 minus a thing", and by verbal argument ("rhetorical algebra") they would transform it into the standard problem "10 things are equal to 21 dirhems and a square", the solution of which was known from a standard algorithm. Diophant would speak abstractly of two numbers, the sum and product of which were given (Arithmetica I, xxvii - Tannery 1893:I, 60f; Ver Eecke 1926:36); he would exemplify the method in a concrete case, "sum equal to 20 units, product equal to 96 units", represent the numbers by "1 number augmented by 10" and "10 minus 1 number", and he would proceed until the complete solution by rhetorical methods.

Similar matters are treated in Euclid's so-called "geometrical algebra", in Elements II,5, by strictly geometrical arguments: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half" (Heath 1956:I, 382). The Data on their part contain both the analogue of the original two-number problem (§ 85) and the analogue of the one-number problem to which al-Khwârizmî would reduce it (§ 58) (see Thaer 1962:40,57; cf. Elements VI, 28).

Quite different from this is the sort of geometry used by al-Khwârizmî to justify the standard algorithms given for the solutions of the basic mixed second-degree equations. To avoid any confusion with the much-discussed "geometrical algebra" I will propose the term "geometrical heuristics". Since this concept will be of basic importance in the following, I shall present it more fully.

Let us for instance consider al-Khwârizmî's justification of the algorithm solving the equation "Square and Roots equal to Number", concretely $x^2 + 10 \cdot x = 39$ (Rosen 1831:13ff). Figure 1A shows a rectangle the total area of which is $x^2 + 10 \cdot x$, i.e., according to the equation, equal to 39. This figure, which is only implicit in the text, is transformed in one of two ways. One possibility is to cut the rectangle $10 \cdot x$ into four rectangles of each $\frac{10}{4} \cdot x$, which are placed along the edges of the square x^2 (Figure 1B). This cross-formed surface is completed by means of four squares of each $\frac{10}{4} \cdot \frac{10}{4}$ (dotted lines); the result is a square, the area of which is $39 + 4 \cdot (\frac{10}{4})^2 = 39 + (\frac{10}{2})^2$, and whose side is therefore $\sqrt{39 + (\frac{10}{2})^2}$. Since, at the same time, the side is $x + 2 \cdot (\frac{10}{4}) = x + \frac{10}{2}$, the solution $x = \sqrt{39 + (\frac{10}{2})^2} - \frac{10}{2}$ is justified.

The other justification, less symmetric but more direct, is shown in Figure 1C. The rectangle $10 \cdot x$ is cut into two rectangles of each $5 \cdot x$, the area $x^2 + 10 \cdot x = 39$ is rearranged as a gnomon and completed by means of a square $5 \cdot 5$.

Both procedures can be described as "cut-and-paste" procedures, involving partition, rearrangement, and completion. Both are clearly non-Euclidean in character, with respect to the conceptualizations involved and as to the rigorousness of the "demonstration", which by al-Khwârizmî consists in "seeing" - one could speak of "intuitive justification" or "geometrical heuristics".

In contradistinction to all the algebras discussed above, Babylonian algebra, their common ancestor, is traditionally claimed never to tell its methods but only its algorithms. And indeed, if we look at a variant of the problem x+y=a, xy=b, viz. BM 34 568, problem 9 (MKT III, 15, 17), it looks very much so[1]:

> Length and width added are 14, and 48 the surface.
>
> They are not known. 14 times 14: 3`16'. 48 times 4: 3`12'. From 3`12' you go up to 3`16' and 4 remain. What times what shall I take to get 4? 2 times 2: 4. From 2 you go up to 14 and 12 remain. 12 times 30': 6. 6 is the width. To 2 you add 6: 8. 8 is the length.

In modern language:

$x + y = 14$; $x y = 48$.

$14 \cdot 14 = 196$; $48 \cdot 4 = 192$; $196-192 = 4$;

$\sqrt{4} = 2$; $14-2 = 12$; $\frac{1}{2} \cdot 12 = 6 = y$; $2+6 = 8 = x$.

This text represents an extreme. Often the texts will identify the numbers used in the single calculation by pointing to their origin or to a point in the calculation where they were already used (in a text like the above e.g. "to 2 the square-root you add 6 the width"). A few texts go even further and formulate in abstract terms the algorithm to apply. So BM 34 568, problem 18 (MKT III:16f,19)

Length, width and diagonal added: 1`. 5`the surface. Length, width and diagonal with length, width and diagonal multiply. Surface with 2 multiply ...

In modern language:

$x + y + d = 60$; $x y = 300$

$(x + y + d) \cdot (x + y + d)$; $2 \cdot xy$

$(x+y+d) \cdot (x+y+d) - 2 \cdot xy$; $\frac{1}{2} \cdot ((x+y+d) \cdot (x+y+d) - 2xy)$

$d = \frac{1}{2} \cdot ((x+y+d) \cdot (x+y+d) - 2xy)/(x+y+d)$.

Even this opposite extreme is, however, nothing but a more abstract description of an algorithm, and tells nothing about algebraic conceptualizations.

Now, obviously, the algorithm which al-Khwârizmî justifies by means of geometrical heuristics could just as well be justified by modern symbolic transformations of the equation and, indeed, by rhetorical methods or by the application of Elements II.6, as was done by Thâbit ibn Qurra (see P. Luckey 1941:105f). No sequence of arithmetical calculations betrays the pattern of mathematical thought by which it was generated. The only thing of which we can be fairly sure is that some representation was present (physically or mentally) to the mind of the author which gave a meaning to at least most of the steps of the calculation - few persons could plan a calculation of seven steps and nobody could teach it without giving reasons why the single calculations were to be performed.

So, Thureau-Dangin was on thin ice - as should be the first explorers - when claiming that the scribe did not formulate a certain equation "but certainly had it in view" because he performs certain calculations (TMB p. xx). When van der Waerden (1975:71f) proposed the use of geometrical heuristics as the source for such knowledge that $(a+b) \cdot (a-b) = a^2-b^2$ and $(a+b)^2 = a^2+b^2+2ab$, he was definitely more sober-minded when advancing his suggestion only as a conjecture.

The problem, how to infer from the results and algorithmic procedures of Babylonian algebra to its methods, concepts, and patterns of thought, seems unsolvable.

THE LITERAL READING: A SOLUTION?

It seems, but maybe it isn't. Indeed, when describing their algorithms the Babylonians used a vocabulary which was to a large extent but neither fully nor always technical. Closer analysis of the formulation of certain texts appear to reveal traces of a description of method. Maybe they might have shown much more, had not our only key to the meaning of the technical vocabulary often been the numerical calculations in which the technical terms occur.

Many of the suggestive formulations point towards the use of some sort of geometrical argument. On the other hand, the real geometrical texts demonstrate beyond doubt that rigorous geometry of the Greek brand was not known to the Babylonians. So, it seems reasonable to approach the texts with the working hypothesis that geometrical heuristics might not have been the invention of al-Khwârizmî or his contemporaries but could belong to the age-old algebraic tradition, going back perhaps to its Old Babylonian origins,- and many texts, indeed, offer a positive response to this approach.

Cf. ***addendum*** *p. 104*

[Originally, this page contained a discussion of the text IM 52.301. edge, which, if an interpretation due to Bruins (1953:242f, 252) were correct, would possess close affinity with al-Khwârizmî's ***first*** *justification of the algorithm for the "square plus sides"-problem. However, as demonstrated by Gundlach & von Soden (1963:253, 255, 259f), the text should in all probability be read quite differently, and has nothing to do with geometrical treatment of arithmetical relations. Cf. also the appendix, pp. 105.45 and 105.53ff.*

Let us first consider the second part (line 8 onwards) of a commercial problem from the Old Babylonian period of Susa, Texte XIII in TMS (pp. 82ff) - below, we shall return to the first part and justifie the point from where we start:

Two quantities x and y, $x>y$, fulfill the relations $x-y=4$, $x\cdot y=1`17'$ (x is the number of qa of oil bought for 1 šekel of silver, y the number of qa sold for the same weigth; so, since silver fulfilled the monetary function, x and y are the "prices", more precisely the reciprocals of the prices of purchase and sale). In the standard formulation, this problem would run "the surface [of a rectangle] is 1`17', the length over the width 4 goes beyond".

The procedure is described as follows[3]:

8. ½ of 4 break off (hipi), 2 you see (tamar, from amārum, "to see", "to meet with"). 2 turn [into a frame, i.e. "square"] (NIGIN[3a] saḫārum, "to turn oneself", with connotations of "periphery" and "to surround"), 4 you see.
9. 4 to 1`17' append (daḫ~waṣābum), 1`21' you see. How much the side (íb-si)? 9 the side.
10. 9 for appending pose. ½ of 4 which you cut away [from the "price"] break off, 2 you see.
11. 2 to 9, the first, append, 11 you see, from 9, the second, you tear off (zi~nasāḫum),
12. 7 you see. 11 qa for each [šekel] you have bought, 7 qa you have sold.

Addendum: *"for appending" (daḫ) should be DUḪ, "its equal", "its identical copy" (when used of tablets), or eventually "its opposite" - see Gundlach & von Soden 1963:261 (other improvements of the readings are also found here). On the term DUḪ, cf. this paper, p. 93 (line 40), n.5 and n.35.*

Line 10: "Cut away" translates kašāṭum - see AHw I, 462a.

This is easily followed on a figure. Moreover, it can be interpreted as a description, step by step, of a geometric procedure - see Figure 3. In the rectangle AC (Figure 3A), AB represents the length (x), AD the width (y), EB that which was cut away from the "price", or, to use the current expression from the standard problems dealing with rectangles, "so much as (mala) the length (uš) over (eli) the width (sag) goes beyond (iteru)". Accordingly, AE=y.

From EB (and from the corresponding part of the rectangle) one half is, literally, broken off, and this half, viz. FC, is placed along DG as DK (Figure 3B). The half is squared, by an expression NIGIN which suggests a turning movement and/or the the creation of the square frame GHJK, and appended to the gnomon arising from the transformation of our original rectangle In many of the texts which we shall meet below, the squaring would be described by the term šutamhurum, which literally involves reciprocity, "raise equals/equivalents against each other" (GAkGr § 92^{c-d}, 94^{d}). By appending we get the complete square AJ (one will notice the agreement with Figure 1C), of area 1`21'. The square-root is found, i.e. the side AL=AF of this square. The square-root is referred to by a Sumerian expression which involves the concept of one thing being equal to another one. The Akkadian equivalent used in paralllel texts (mitḫartum) (related to the šutamḫurum mentioned above) means "a thing which stands against its equal". The term might of course have become a technical one, involving no such connotations, but line 11 shows that the square is indeed thought to have two different square-roots - so, a more adequate alternative to the translation "square-root" would be the expression "side of the square".

Cf. also the corrected line 10.

Finally, the breaking-off ½ from the price-difference (FB) is repeated. It is added to the 1st[4] side of the square (AF), yielding AB=x; and it is subtracted from the 2nd side (AL), yielding AD=y.

A number of points in the above should be noticed:

- Bisection of a quantity is expressed concretely, as "breaking-off", not (as in the Seleucid problem discussed on p. 4) as a multiplication by 30′ - this will have implications for a possible understanding of the development of Babylonian algebra.

- Squaring is, even in Akkadian which in the Old Babylonian period was new as a mathematical language, expressed by a term which suggests turning and/or creation of a frame, as found in geometrical squaring. Reversely, a quantity possesses not one but two equal "square-roots" or "sides", not only in the non-technical connotations of the Sumerian as well as the Akkadian terms but even unmistakeably in the formulation of the solution.

- The distinction between the "1st" and the "2nd" side of the square is such as to give meaning geometrically. This is no matter of course. Indeed, we should notice that the 2 (half the price-difference) which are added to and subtracted from the 1st and 2nd side of the square, respectively, is one and the same quantity - the number stands as the un-repeated object of the sentence. Had instead the text spoken of adding the 1st half (e.g. FB) and subtracting the 2nd half (e.g. EF) from a side (singular - e.g. AF), we would have been unable to find the results as length and width on a geometrical figure. This consequently restricted use of the distinction between "1st" and "2nd" "half" or "side" holds throughout the lot of problems where I have analyzed its occurrence. If no geometrical representation was present (physically or mentally) to those formulating the texts, i.e. if nothing but the idea of an arithmetical splitting of a quantity into addends was expressed, then no reason exists that such consequent behaviour should have arisen.

- Finally, one will observe that an active procedure of construction is described in the text. If geometrical heuristics is described, then the figure used is created by the "you" of the text, no pre-existing figure is used as the foundation of the argument (even though, surely, the construction follows a pre-existent pattern).

A closely parallel problem, the formulation of which throws supplementary light on the above, is YBC 6967 (MCT, p. 129f). It deals with two quantities igibûm and igûm, a number and its complement in the table of reciprocals. In this case, their product should be understood not to be 1 but 1˴- i.e., both entries in the table are understood as belonging to the first order of sexagesimal magnitude (a circumstance which suggests that the terms should be understood primarily with reference to the tablet, not as designating abstractly reciprocal entities):

Obv.1. The igibûm over (eli) the igûm 7 goes beyond (itter),
2. igûm and igibûm what?
3. You,7 which igibûm
4. over igûm goes beyond
5. to two break (ḫipi): 3˚30′.
6. 3˚30′ together with 3˚30′
7. give reciprocally (šutākil- further explanation below): 12˚15′.
8. To 12˚15′ which went out for you
9. 1˴ the surface (eqlum) append (ṣib, from waṣābum): 1˴12˚15′.
10. The side (íb-si₈) of 1˴12˚15′ what? 8˚30′.
11. 8˚30′ and 8˚30′ its equal[5] (meḫrum) lay down (idi, from nadûm, to lay down, also in a drawing)

Rev.1. 3˚30′ the thing to which was "given" (in line obv.7 - takīltam[5a])
2. from (ina) one (ištēn) tear off (usuḫ),
3. to (ana) the other (ištēn[6]) append
4. the first (ištēn) is 12, the second (šanûm) 5.
5. 12 is the igibûm, 5 the igûm.

Even this can be followed on Figure 3. Important to observe in this case are the following points:

- The formation of the square GJ (3˚30′ times 3˚30′) is this time described by the term šutākulum, the causative, reflexive derivative of akālum or, perhaps, of kullum. The first, philologically more plausible stem, means "to eat" and, from there, "to receive", "to receive the benefits of"; this stem would make the derivative mean "to make

Addendum: According to von Soden 1964:50, the derivation from kullum is improbable, since kú, used ideographically for akâlum, is also *used as an ideogram for šutākulum (cf. also MEA p. 55, no. 36)*

To avoid misunderstandings: The choice of "giving reciprocally" is not intended to imply that this is in any semantic sense the meaning of šutākulum - in fact, the words when taken together thus are rather meaningless. The expression is chosen to avoid any translation which would carry a precise but dubious or anachronistic meaning. A hieroglyphic expression "to make x and y XX each other" might do quite as well. But to speak of "multiplication", as do the dictionaries, would, in the context of this investigation, be to beg the point, and furthermore to beg the point in a misleading direction (since, as it will turn up, the Babylonians distinguish several different multiplications of which šutākulum is only one).

receive one another". Kullum, "to hold", would make it mean "to make hold one another" (i.e. in a right angle). The latter, geometrically suggestive interpretation was proposed by Thureau-Dangin (1937a:23 n.1). As a term which expresses the information contained in the grammatical form without being too clumsy and without favouring any of the possible alternative stems, I have chosen "to give reciprocally" [in order that the two may either receive or hold one another]. In any case, squaring is seen, as when described by the terms NIGIN or šutamḫurum, as a reciprocal act involving two equal quantities, not as a single look into the table of squares. Truly, even the tables of squares list "n times n" (see MKT I, 69f), but the connector "times" (a-rá, derived from the concept of "step" or "going", see ŠL II[4] no. 579,237b, cf. II[2] no. 206,5) has no relation whatever to neither NIGIN, šutākulum or šutamḫurum. So even though the starting-point of the problem was one of the standard tables, the reference to such a table is glaringly absent from the description of the solving procedure.

- Once again, the square-root is taken twice, in a language which more or less enforces a locational interpretation.

- Even the additive and subtractive operations ("append to"; "tear off"; "go beyond") have geometrical connotations.

These two texts both refer to the same geometric construction (if they refer to one at all), and to one described by al-Khwârizmî at that. We shall next consider a problem type which in Babylonian algebra was treated in a way characteristically different from that of the Medieval mathematicians, viz. $ax^2+bx=c$.

BM 13901, problem 3 (MKT III, 1, 6) runs as follows:

9. The third of the surface I have torn off (assuḫ, from nasāḫum), then the third of the side [of the square] (mitḫartum, "the thing which stands against its equal", cf. p. 8) to inside (ana libbi)
10. the surface I have appended (uṣib, from waṣābum): 20′.1 the wāṣitum[7)] you pose (tašakkan, from šakānum)
11. the third of 1 the waṣitum, 20′, you tear off: 40′ to 20′ you raise (tanašši, from našûm, "to carry", "to lift up", "to raise", from where "to multiply").
12. 13′20″ you inscribe (talappat). The half of 20′, of the third which you tore off [error for "which you appended]
13. you break off (teḫippi), 10′ and 10′ you give reciprocally (tuštakkal, cf. p. 11), 1′40″ to 13′20″ you append,
14. 15′ is, 30′ the side (ib-si₈). 10′, which you gave reciprocally, inside 30′ you tear off: 20′.
15. The reciprocal of 40′ (igi 40 gál-bi, the formulation of the table of reciprocals), 1°30′, to 20′ you raise: 30′ the side [of the square] (mitḫartum).

This may be a bit difficult to follow. In symbolic algebra, what goes on is the following:

$$(1-\tfrac{1}{3})x^2+\tfrac{1}{3}x=\tfrac{1}{3}\ :\ \tfrac{2}{3}\cdot\tfrac{2}{3}x^2+\tfrac{2}{3}\cdot\tfrac{1}{3}x=\tfrac{2}{3}\cdot\tfrac{1}{3}\ ;$$

$$\tfrac{2}{3}\cdot\tfrac{2}{3}x^2+\tfrac{2}{3}\cdot\tfrac{1}{3}x+(\tfrac{1}{2}\cdot\tfrac{1}{3})^2=\tfrac{2}{3}\cdot\tfrac{1}{3}+(\tfrac{1}{2}\cdot\tfrac{1}{3})^2=a;\ \tfrac{2}{3}x+\tfrac{1}{2}\cdot\tfrac{1}{3})=\sqrt{a};$$

$$\tfrac{2}{3}x=\sqrt{a}-\tfrac{1}{2}\cdot\tfrac{1}{3}=b;\ x=(\tfrac{2}{3})^{-1}\cdot b$$

This procedure is remarkable: Indeed, Arabic algebra would at once divide by the coefficient of x^2, reducing thereby the problem to a standard type. We might say that the Babylonians preferred to look at the problem as dealing with the quantity $\frac{2}{3}x$.

For any rhetoric way of thought, this is a clumsy method. Anyone who has read Medieval algebra will know that the rhetoric method makes you prefer one fixed variable together with the standard names for its powers. Yet, if we try to construct a geometrical heuristic figure in accordance with the problem, the advantage of the Babylonian method at once stands out - see Figure 4:

Addendum: Truly, Diophant follows the Babylonian procedure in spite of his rhetorical methods (VI,vi, see Tannery 1893:I,420, cf. Vogel 1936:714), and Heron does almost the same (Geometrica 21, 9-10, Heronis Alexandrini opera IV, 380[15ff], 381[13ff]). However, the closeness of both to the Babylonian tradition may explain this use of a procedure which does not fit the rhetorical method optimally.

To a problem $ax^2+bx=c$, a rectangle is imagined with sides $a\cdot x$ and x (a=2/3), and to this is appended a rectangle $b\cdot x$. This is shown on fig. 4A, where also the full square x^2 is shown (dotted line).

Fig. 4B shows the multiplication by a. The total area is now ac. By the normal cut-and-paste-procedure, this is transformed into the gnomon of Figure 4C, which is completed by the small square $(\frac{b}{2})^2$, etc. When, in this way, the quantity ax has been found by the now familiar procedure, the original side of the square is found by multiplication by a^{-1}.

Should instead the reduction have followed the Arabic procedure, the transformation of the appended rectangle $b\cdot x$ would inevitably have been graphically more difficult, as it would necessitate a change of width of the rectangle $b\cdot x$.

Apart the geometrically suggestive terms for a number of arithmetical operations now familiar (including now also the term ana libbi, "to inside", from libbum, "heart", "bowels", the prepositional and adverbial form of which, libbi, suggests that the entity governed can be considered a totality,"a body") we notice this time that the two changes of scale (the multiplication by 40′ in line 11 and the multiplication by 1°30′ in line 15) are expressed neither by means of the usual geometrically connotative vocabulary nor in the term known from the multiplication tablets (a-rá, "step"). Instead, the seemingly queer expression našûm,"to carry", "to raise".

According to von Soden (private communication), libbi cannot be regarded a real preposition in the Old Babylonian period. So, the use of the term in just this function in the present tablet must have been read by its contemporaries with the basic corporeal meaning inherent as a clear connotation.

Truly, the occurrence in problem 3 is a restitution, but a restitution from other undamaged parts of the tablet where the construction "X libbi Y tanassah" is attested beyond doubt - e.g. obv. II, 31 and 34 (translation below, p. 49).

A clue which appears to make good sense is supplied by the Sumerian equivalent to the term: íl (see MKT II, 29a). In BM 85196, rev. II,11 the term íl[tum] (íl~našûm + ending tum, "the act of / thing {lifting" / being lifted" ?) is used apparently as a measure of slope (the gradient or its inverse) (see MKT II, 46, 52). Now, evidently such a factor is used as a coefficient of proportionality, and the term "raise" suggests that the change of scale was thought of in the analogy of Fig. 5, A and/or B. Both figures explain that multiplication by a factor of scale can be expressed as "lifting" or "raising".

Addendum: It will be remembered that Babylonian geometrical illustrations are not made to scale when they occur (cf. Thureau-Dangin 1897, where a field-plan is shown together with the correct proportions). So, the change of scale in one direction proposed here can easily have been imagined by the Babylonian mathematician; on the other hand, the changes required by the "Medieval" solution would be more difficult to handle mentally.
(cf. p. 105.1ff).

Before looking at other problems from this tablet, we shall consider VAT 8390 (MKT I,335ff), the formulation of which looks very strange until a geometric interpretation is introduced. The first problem runs as follows:

Obv.I.1. Length and width I have given reciprocally (uštākil, see p. 11): 10` the surface.

2. The length to itself I have given reciprocally:

"Build" translates banûm, cf. below, p. 31.

3. a surface I have built.

4. So much as (mala) the length over the width goes beyond

5. I have given reciprocally, until 9 I "doubled" (eṣṣip, from eṣēpum, "to double", even in an extended sense): (it is)

Taken by itself, the clause "which the length by means of itself gave reciprocally", "ša uš ina ramānišu uštakilu", may also be the first person, "which I gave the length reciprocally on/from/by itself". In that case, however, it is difficult to see why the preposition should change from "ana" (obv.I.1) to "ina" when a main sentence is transformed into a relative clause (both constructions occur 4 times in the tablet; in a reasonable stochastical model, the probability that the prepositions should be distributed as they are is $2^{-8} = 1/256$).

Hence, it appears that the uštākilu occurring with ina is Št-I (causative passive), which for akālum coincides with Št-II (the causative reciprocative form used elsewhere).

6. the same as (kima) that surface which the length by means of (ina) itself

7. gave reciprocally (uštākilu).

8. Length and width, what? (ennam).

9. 10` the surface pose (gar-ra~šakānum)

10. and 9 which "doubled" pose:

11. The side (íb-si$_8$) of 9 which "doubled", what? 3.

12. 3 to the length pose.

13. 3 to the width pose.

14. Since (aššum) »so much as the length over the width goes beyond

15. I gave reciprocally«, he said (iqbu, from qabûm)

16. 1 from 3 which was posed to the width

17. tear off (usuḫ, from nasāḫum), and 2 you leave (tezzib, from ezēbum).

18. 2 which you left, to the width pose

19. 3 which to the length was posed

20 to 2 which to the width was posed, raise, 6.

21. The reciprocal of 6 [literally, "part (igi) 6"] find out (puṭur, from paṭārum)]: 10′.

22. 10′ to 10` the surface raise (íl), 1`40`.

23. The side (íb-si$_8$) of 1`40`, what? 10.

II.1. 10 to 3 which to the length was posed

2. raise, 30 the length.

3. 10 to 2 which to the width was posed

4. raise, 20 the width.

The solution is followed by a proof:

5. When 30 the length, 20 the width,

6. the surface, what?

7. 30 the length to 20 the width raise, 10` the surface.

8. 30 the length together with (itti) 30 give reciprocally: 15`.

9. 30 the length over 20 the width how much it goes beyond? 10 it goes beyond.

10. 10 together with 10 give reciprocally: 1`40`.

11. 1`40` until 9 double: 15` the surface.

12. 15` the surface, the same as 15` the surface which the length

13. by means of (ina) itself gave reciprocally.

In symbolic algebra:

$$xy=600 \; ; \; x^2=9\cdot(x-y)^2$$

The next steps are difficult to formulate inside this framework. Approximately, the text puts

$x=3z$; $y=3z$, and corrects (because $x=\sqrt{9}(x-y)$) to

$x=3z$; $y=(3-1)z=2z$.

From here, xy is calculated in terms of z (as $6z^2$), from where $z^2=100$, $z=10$, $x=30$ and $y=20$.

The substitution x=3z could be justified from the fact that x^2 is a multiple of 9. But there is no obvious reason that y should be given a provisional value 3z before being corrected. If not the next problem had been completely parallel to this one, line 13 might have been the product of a sloppy scribe; but there is the making of a parallel, and nothing sloppy in the rest of the formulation. So, the provisional width of 3 requires an explanation,- and it gets one as soon as a figure is drawn (see Figure 6).

In general, every single conclusion discussed below is supported by the parallel second problem.

Figure 6A shows our rectangle (bold line), together with the squares x^2 and $(x-y)^2$. If the latter is contained 9 times in the former, the pattern of Figure 6B proposes itself. On this figure, the large quadrangle contains the small three times in each dimension,- and so, its length as well as its width is measured by the number three. To obtain the width of the original rectangle, the side of one small square is subtracted, and we are left with two.

One possible interpretation of the rest is this: The number of small squares inside the original rectangle is calculated, their area and thereby their side found, and finally x and y calculated.

Another interpretation is this: A model figure (Figure 6C) is drawn or imagined with real length and width 3, and the area of the part corresponding to the original rectangle is found to be 6. Then the coefficients of quadratic and linear scale are calculated

At closer inspection, the formulation of the text favours the first of the two interpretations strongly: While in obv. 10, "10` the surface" is posed, in obv. 12 and 13 (and all following parallel places) 3 is posed <u>to</u> (ana) the length and the width. So, 3 is not considered identical with a length and a width - it is simply a number written <u>to</u> the two dimensions of the pre-existing figure, like the "proportionate numbers written to the upper and lower length in Str. 367 (see below, p. 105.34ff), and like the proportionate numbers 1 and 5 which, in the partially analogous YBC 4608, obv. 5 (MCT, p. 49), the student is asked to pose "by" (<u>ina</u>) the upper and the lower width. (The numbers are proportionate numbers for the lengths, not the widths by which they are posed; so, it is out of the question that the numbers 1 and 5 might be "posed as" widths; at the same time, the "posing by" the widths is repeated so often and with such consequence in the text that a simple writing error is excluded).

1`40' and 10, respectively), etc. The use of the multiplicative term "raise" (íl ~ <u>našûm</u>) in obv. I.22, obv. II.2 and obv. II.4 might look as a support for such an interpretation, but since the term is also used for multiplications which have nothing to do with changes of scale (obv. I.19f and obv. II.7), this support is illusory (cf. also a closing discussion of the terms "raising" and "giving reciprocally" below, p. 83).

Regardless of the choice between the two possibilities, a geometrical interpretation is supported by the use of the term "surface", a-šà, for three different entities which all play a role as real surfaces in the geometrical solution (obv. I.1 and 3; obv. II.6f, 11 and 12). The same can be said on the use of "width", sag, for two different entities which, geometrically seen, are both real widths (e.g. obv. I.13 and 18).

Terminologically, two observations could be made, both concerning the term "reciprocal giving" (<u>šutākulum</u>).

Firstly we see that this term is not used in all cases where a rectangular (including square) area comes about. As it is elucidated by a parallel text discussed below (p. 83), "reciprocal giving" seems to be reserved to cases where a rectangular surface is <u>made</u>, while "raising" describes the simple calcula-

Cf. also p. 38, the marginal note.

tion of the area of a surface which is already there.

Secondly, multiplication by "reciprocal giving" is expressed sometimes with the factors as the objects and sometimes as the subjects of the act of giving. It is sometimes symmetrically expressed, by the preposition "together with" (itti), sometimes asymmetrically, by "to" (ana - when one factor is the object of the act) or by "of"/"in"/"on"/"by means of" (ina - when one factor is the subject and the surface constructed is the object)

The latter observation suggests a more general conclusion: The author of our text has no fully technicalized mathematical vocabulary at his disposal. He describes his procedures as best he can, in an only partially conventionalized language (cf. further evidence in this direction below, p. 38). Language can therefore hardly be the basic medium for what goes on. Instead, language describes some other conceptual or physical representation of the quantities and manipulations involved. (So, the fluidity of the Old Babylonian mathematical phraseology should reflect itself in a translation which tries to render the cognitively significant aspects of Old Babylonian mathematics).

Cf. also the marginal note below, p. 64, on the variety of rare or once-occurring subtractive terms, and p. 105.15, the marginal note on muttarittum.

We shall now return to BM 13901. Problem 10 (MKT III,2f) can, together with problem 14, elucidate the open question of the above, "Figure 6B or Figure 6C?" The text runs as follows:

Obv.II.11. The surfaces of my two square figures (mitḫaratija) I have taken together (akmur, from kamārum, "to pile up", "to accumulate"): 21˚15′

12. (from) figure (mitḫartum)to figure by a seventh it became less (imṭi, from maṭûm, "become small(er)").

13. 7 and 6 you inscribe (talappat, from lapātum). 7 and 7 you give reciprocally (tuštakkal), 49

14. 6 and 6 you give reciprocally, 36 and 49 you take together (takammar, cf. l. 11):

15 1`25˚. The reciprocal (igi) of 1`25˚, however, is not found out.
What to 1`25˚

16. should I pose (luškun, from šakānum) which 21˚15′ gives me (inaddinam, from nadānum)?
15′, 30′ the side (íb-si$_8$).

17. 30′ to 7 you raise: 3˚30′ the first side [of square];

18. 30′ to 6 you raise: 3 the second side [of square].

The text of no. 14 runs like this:

Obv. II.44. The surfaces of my two square figures I have taken together: 25`25′.

45. (one) figure two thirds of (the other) figure, and 5 GAR (the normal measure of length, corresponding to the side of the unit area mentioned in note 7, the sar).

46. 1 and 40′ and the 5 beyond the 40′(elenu 40′, cf. eli in the expression "over ... goes beyond") you inscribe.

47. 5 and 5 you give reciprocally, 25 inside (libbi) 25`25' you tear off:

Rev.I.1. 25` you inscribe. 1 and 1 you give reciprocally,
1. 40′ and 40′ you give reciprocally,

2. 26′40″ to 1 you append: 1°26′40″ to 25` you raise,

3. 36`6°40′ you inscribe. 5 to 40′ you raise: 3°20′

4. and 3°20 you give reciprocally, 11°6′40″ to 36`6°40′ you append,

5. 36`17°46′40″, 46°40′ the side (íb-si$_8$). 3°20′ which you gave reciprocally

6. inside 46°40′ you tear off: 43°20′ you inscribe.

7. The reciprocal (igi) of 1°26′40″, however, is not found out. What to 1°26′40″

8. shall I pose (luškun, from šakānum) which 43°20′ gives me? 30 its ba-an-da (a term the meaning of which can only be disclosed from the context - mathematically, a "factor required" must be meant").

An assimilation of ba-an-da to ba-a, ½=30′, could be tempting; it is, however, ruled out by Rev. I.35, where a ba-an-da of 15 occurs.

9. 30 to 1 you raise, 30 the first side [of square];

10. 30 to 40′ you raise: 20, and 5 you append:

11. 25 the second side [of square].

Before going to the discussion of these texts, still another text should be at our disposal: BM 15285, problem 10 (MKT I,138; a new investigation of the tablet after another fragment has been found changes nothing of importance, see Saggs 1960:139):

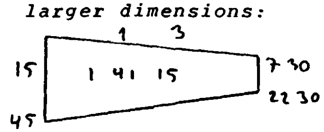
1. 1 the length, a square figure (mitḫartum).
2. In its inside (libbušu), 16 of a square figure (mitḫartim, genitive singular)

3. have I laid down (adi, from nadûm). Its surface what?

These lines stand below a drawing (see Figure 8), in a tablet where numerous such drawings of geometrical figures inside squares are found together with similar descriptions and questions. In this whole tablet, the term nadûm, "to lay down", has the clear meaning "to draw", and the derivations from libbum,"heart", means "inside" as I translated provisionally in the above when it occurred prepositionally.

If we look now on BM 13901, no. 10, it deals with the sum of the surfaces of two squares, of which one (i.e. its side) is less than the other by one seventh. If we follow the interpretation corresponding to figure 6B, the greater side is divided into seven parts, and the smaller into six of the same length. The numbers of small squares are added, the area of each small square and then its side is calculated, etc. If we follow 6C, two auxiliary squares with sides 7 and 6, respectively, are drawn, their total area is found, and thereby the scale (for both interpretations, compare Figure 7).

In YBC 11,126 (MCT p. 44), two sets of numbers are inscribed on the same trapezoid, of which one is three times the other (and an area corresponding to the larger dimensions:



($45 = 3 \cdot 15$; $22°30′ = 3 \cdot 7°30′$; $1``41`15° = 3` \cdot \frac{45 + 22°30′}{2}$)

This would suggest the point of view of 6B. In any case, it shows the application of "scaling in two dimensions" to a genuine geometrical figure.

Cf. also the discussion of the term makṣarum in the appendix, p. 105.11f.

From the point of view of modern algebra, this makes no difference[7]. When $x = 7z$, you get the same operations whether you call 7 or z the scaling factor. Terminologically too, we seem to be in a blind alley. Even though the expression "raise a to b" is not symmetric.

7 (but still, elementary arithmetic teaching upheld the completely corresponding distinction between "measuring" and "dividing" until recent times.

I have not succeeded in finding sufficient consequence in its application to decide whether a or b in such an expression is the scaling factor; furthermore, when the area of the small square is found (interpretation "6B"), or when the factor of quadratic scale is found (interpretation "6C"), both by solving the problem Y·1`25`=21`15′, then this multiplication is formulated by the semantically neutral[9] <u>šakānum</u> ("pose [for action upon]"), normally used for divisions by irregular numbers (i.e. numbers with no finite reciprocal in the sexagesimal system). The analogous passages of no. 14 give us no supplementary clues.

However, the short text from BM 15285 provides at least a hint. Here, it is clearly an existing square which is subdivided, and so the cognitive suppositions of interpretation "6B" are seen to be present. Furthermore, it seems conceptually simpler to ask, when their sum is known, for the area of one of 49+36 equal squares with a geometric interpretation like that of Figure 8, than to formulate the question of quadratic scale. So it seems safe to follow Ockam's principle and assume that the Babylonians followed the one path to which we know they had access, i.e. "6B", and not the other one which theoretically might have been used.

Maybe, however, the whole discussion is, after all, superfluous: The symmetrical use of the asymmetric term <u>našûm</u>, "raise", should perhaps be taken as an indication that the mathematical equivalence of the two conceptualizations had become trivial to the point that the difference became invisible.

Terminologically, we notice the following:

- <u>mitḫartum</u>, "that which stands against its equal", which in other texts I translated as the "side of the square", must in BM 15285 no. 10 (and other texts of the same tablet) necessarily be translated as a "square" or, better, "a square figure"[9a], a term which avoids our vague Euclidean connotation according to which a square is a surface and not its border (a clumsy but perhaps still better term would be "a square frame"). In BM 13901 no. 10, once more we find in line 11 that a <u>mitḫartum</u> is something to which <u>corresponds</u> a surface, while in line 12 we see that it is <u>identified</u> with its linear extension. So, the Babylonian <u>mitḫartum</u> is not a term which is used in two different senses, "a square" and "a square root" or "side of the square"; instead, it signifies a single concept which happens to be incongruent with ours (the frame producing the surface and not the surface produced), and which therefore can find no adequate single translation into modern geometrical idiom. Incidentally, we can notice what seems to be a parallel to the identification of "side" and "square figure": When, as we have often seen, a rectangle is broken into halves to allow the creation of a gnomon, this is referred to as a bisection of the side; the case of the square shows that this way to speak constitutes no objection to the geometrical interpretation of this bisection, but is a simple consequence of a general pattern of thought.
- the additions of the two surfaces and of the numbers of small squares are expressed by a symmetric term, <u>kamārum</u>, "to pile up". The same term is used e.g. in no. 1 where the surface of square and the side are "taken together" (<u>eqlam</u> <u>u</u> <u>mitḫarti</u> <u>akmur</u>). Addition expressed by <u>waṣābum</u>, "to append", occurs only in

Addendum: *In TMS texte VI (p. 49),* NIGIN *(otherwise semantically related to šutamḫurum and šutakulum, cf. note 3a) is used for a square figure, which has both an explicitly designated area (*a-šà*) and an explicitly designated side (*uš*). In texte V, § 10 (p. 42),* LAGAB *is used in the same way, while in § 5 and 6 it designates the side of a square. So, the conflation of "square figure" and "side of square" as well as the semantic relationship between this conflated concept and "rectangular multiplication" is obvious.*

An analogous conflation of the circle and its periphery is obvious in BM 80209 (discussed in Friberg 1981), where gúr *stands for the periphery in Obv.5-9 and Rev 1-9 (a periphery which is understood as the circular figure, since it possesses an area), while in Rev.10-13 it is the name of the figure alone, the perimeter of which is designated siḫirtum of the* gúr *(from saḫarum, cf. note 3a).*

So, the understanding of plane surfaces as emanations from a linear extension is recurrent and must be considered a basic feature of Babylonian mathematics.

asymmetric situations where one main quantity is extended by another one, i.e. where the first quantity can be said to conserve its identity while growing in magnitude.

BM 13901, problem no. 14, is also concerned with two squares but one step more complicated. Expressed symbolically,

$x^2+y^2=$25`25` $y=\frac{2}{3}x+5$

The solution follows this pattern:

$x=1\cdot z$, $y=\frac{2}{3}z+5$; then, with $\frac{2}{3}=40'$,

$(1^2+40'^2)z^2+2\cdot(5\cdot 40')z=$25`25`$-5^2=$25`.

This is solved as the corresponding problem of no. 3 (see p. 12f), and $z=30$ is found. Then $x=1\cdot z$, $y=40'\cdot z+5$ are calculated.

The geometrical heuristics which may have been used instead of this anachronism is seen on Fig. 8. The two square figures are shown on Fig. 8A (bold line). Another square is then imagined, the side of which is equal to the first side, and the accumulated area of the two original squares is calculated in terms of surface and side of this new square. The first original square provides 1^2 surface. The second original square provides:

- $5\cdot 5$ which are immediately torn off the total area.
- $40'\cdot 40'=26'40''$ times the surface of the new square.
- [2 times] $5\cdot 40'=3'20''$ times the side of the new square.

This can be drawn or imagined as in Figure 8B, which is for further treatment expanded in vertical direction to become a "square plus sides" cf. Figure 4A-B, and cut and pasted as usual. From this, 1`26'40'' z is found, and next z.

Then the original sides are found by multiplication by 1 and 40', respectively (and it is partly this multiplication by 1, partly the contribution of 1^2 [new] surface instead of just 1 surface which justifies the claim that another square figure is thought of).(Cf. also a discussion on pp. 90f).

This procedure calls for three remarks. For one thing, we see that the auxiliary square is defined directly in terms of the unknown squares, in agreement with the interpretation "6B" of the above. We might try to formulate in agreement with "6C". Then the auxiliary square would be one of side 1 GAR, and another one of side 40' GAR plus 5 GAR would have to be imagined. It is not a priori excluded that such a formulation may have been used - but if this be the case, then only as an insufficient description of a geometrical figure, on which was reasoned in a way which clearly distinguished two sorts of GAR's, real and imagined. This, however, leads back to a description very close to the above.

Secondly, we notice that the total number of "new sides", 2·3`20', is not calculated, because anyhow it should be split into two afterwards. This demonstrates that the treatment of the problem is not one leading up to a standard problem which was solved mechanically, and which was only recognized when followed precisely. The current observation that

Babylonian algebra solved its problems byr reduction to standard problems is only true in a loose sense, e.g. if "standard problems" are understood as heuristic standard representations.

Finally, we see that not all of the procedure can be a cut-and-paste-heuristic. Most of what goes on, viz. the calculation of the "coefficients" corresponding to Fig. 8B, must be regarded as a keeping of accounts. The role of the geometrical heuristic is then: partly to provide the "heads" of the columns ("how many new surfaces"; "how many new sides"; "how much total area"), closely corresponding to accumulation in the coefficients of the various powers of unknowns in symbolic algebra; partly to support the calculation of the single contributions to the columns, e.g. the transformation of the second original square into 5^2, $2\cdot(5\cdot40')$ new sides, and $40'^2$ new surfaces.

Addendum: Below (pp. 87,90) we shall meet this type of coefficient as something which is explicitly spoken of ("as much as there is of sides", etc.). It should of course be distinguished sharply from another concept which is also translated "coefficient", the IGI.GUB of the "tables of coefficients". The two have nothing in common except the translation and their multiplicative role in calculations.

The precise formulation of BM 15285, problem 10, line 2 demonstrates the "heading"-character of the geometrical concepts: Inside the great square figure we do not find "16 square figures" but "16 of a square figure". See above, p. 21, and below, pp. 87 and 90, on the expression "as much as there is of X", kīma X-[im].
Saggs' assumption, that the singular form (which is indubitably to see in the autography) is a writing error (1960:139), seems to be ruled out by the last line: a-šà-bi en-nam, "its surface what", with the singular suffix -bi.
Cf. GAkGr § 139i.

In other, similar problems involving up to four squares this accounting aspect of the problem becomes even more important. Already in no. 10, the accumulation of a total of 49+36=85 small squares can be seen as such an accounting. Any algebra which treats such problems needs a system according to which names can be provided to the "columns". Babylonian algebra, with its predilection for problems of two or more unknowns, would not

be well served by the Greek and Medieval system of "number", "square", "cube", etc. A geometric representation does much better.

For this accounting function the actual drawing is quite superfluous - all that is needed is a conceptual kit providing something of which to think and speak. The same holds, mutatis mutandis, for geometric figures and transformations like those of Figure 2-7: Once you are habituated to this way of thinking you can imagine the drawing without making it. This is a possible explanation why the only traces of intermediate calculations and argument in normal algebraic problem-tablets are occasional traces of numbers on empty parts of the tablets - so on VAT VAT 8390, where most of the numbers written really are numbers which occur in the calculation (and most of these are among the numbers "posed") - see MKT I, 336 n. 1, and MKT II, Tafeln 25 and 50. Only one number, an indubitable 1,30 (90? 1½?) seems quite out of place.

Yet, after all, the number of tablets containing traces of actual performance is too restricted for us to believe that they really present us with the medium of mathematical reasoning. This medium seems to have been varied. Some texts ask the performant to keep intermediate numbers in his head (rēška likil, "that your head retain" - see examples pp. 79 and 80; for further references, see MKT II, 22, rēšu); others ask for quantities which in our geometric interpretation should be drawn as the sides (nadûm) of a square, to be "laid down", using a term which in BM 15285 really meant "to draw a geometric figure" - one instance of this was quoted from YBC 6967, obv. line 11 (p. 11). Occasionally, lapātum, "to touch", "to inscribe", occurs. Most texts, finally, restrict their indications to the seemingly neutral šakānum ("to pose [for action upon]"), which however may be less neutral and in reality connotatively related to "drawing" or "inscribing", cf. n. 9.

According to Tanret, Akkadica 27 (1982), 49, the teacher would also make the drawings of the signs belonging to the "Silbenalphabet A" and the "Syllabar a" "dans le sable de la cour" at the initial level of the scribal education.

Since, however, much evidence of the most varied sorts - as we have seen - points toward the actual use of geometrical heuristic for the solution of various second-degree problems, we may provisionally assume that some drawings were made at some times, at least as a pedagogical means. Not least concerning mathematical concepts the old empiricist aphorism comes true, that nihil in intellectu quod non prius in sensu. And since no drawings have been found except such that illustrate the formulation or the final result of problems (such as those of BM 15285 and the numerous partitions of trapezoids and triangles), we may perhaps also conclude that these auxiliary drawings were not made on the clay tablets.

We need not look very long for possible mediae in which drawings could have been made and which would leave no material traces. The Greek use of sand for this purpose is only known from anecdotes and other literary sources. More important is perhaps the probable etymology of the word abacus, which D.E. Smith derives from the Semitic root ᶜbq, אבק, "dust" (1958:II,156). Indeed, the dust abacus was known in classical Antiquity and used also for drawing geometrical figures. So, the fact that the Arabs seem to have got it from India (or Persia? - their term, "takht", is Persian) does not prove that it was Indian in its first origin.

The whole development of the dust- and counter-abacus is most intricate. It may also be of interest that a 12th century tradition in Western Europe of unknown origin ascribes the abacus to Assyria (here, of course, the question of the connection between the Greek counter abacus and the Middle Eastern "token"-system comes in, as an extra complication).

Cf. Smith 1953:156ff,157f; Tropfke 1980:54,212,235; and Saidan 1978:351f).

Finally, we may take for granted that the geometrical heuristics did not make use of ready-made all-purpose figures hanging in the class-room of the scribal school. All descriptions of procedure prescribe an actual process of construction, the term "you see" (tamar) being only used when the performant can see the results of his own active work (e.g. in the Susa-text quoted p. 7).

VARIATIONS ON THE SECOND-DEGREE THEME

We shall now turn back from these preliminary concluding deliberations and look at other problems of the second degree which will allow us to perceive better the character of Babylonian algebra, and which, especially, will provide variations on the theme of geometrical heuristics.

In order not to make the exposition more heavy than necessary I shall from now on insert fewer Akkadian words into the translations. A key to the Akkadian equivalents of translated terms is found in Table 1, p. 107. Only new, essential expressions will the first time they occur be given parenthetically with the translation. I shall also permit myself to paraphrase or narrate those problems where a precise translation provides no new insights.

A text which permits us to see the problem of geometrical heuristics from a new angle is AO 8862, problems 1-3 (MKT I, 108-111). The tablet belongs with BM 13901 to the oldest algebraic tablets known (see MKT III, 10). All three problems of the tablet deal with rectangles, of which the accumulations of length and width are given together with various second-degree functions of the sides. The following

As far as I can see from those occurrences which I chequed, the terms might perhaps be kimrātum, i.e. fem. pl. instead of fem. sing. The meaning would them be "the things taken together" instead of "the result of a process of taking-together".

Akkadian expressions are new to us:

- kimratum, "accumulation", related to kamārum, "to take together".

- epēšum, "to do", "to proceed", "to undertake"; as a noun, "making", "procedure".

- banûm, "to build", "to manufacture", "to create", "to produce". (Already used above, p. 16).

- šapiltum, "the remaining thing", used about the remainder of a quantity from which something is "torn off" or "cut off". Related to šapālum, "be low", "be small". Also an accounting term.

- ḫarāṣum, "to cut off".

- wabālum, "to bring [to a place]".

- asaḫḫir, "next", literally "I turn around", from saḫārum, to turn oneself, an adverbial marking of the same sort as

- atur, "again", from târum, "turn back".

Cf. GAkGr § 81k.

- la watar, "no further it go!", an indication of a break in the procedure or the exposition - cf. MKT I, 114 n.13. From watārum, "go beyond".

- finally, one will notice the use of the Sumerian expression a-rá, "times", the term of the multiplication tables, as well as the occurrence of the complete phrase of the tables of reciprocals, igi 6 gál, "the sixth part", "the reciprocal of 6".

The text runs as follows:

[Problem 1]

I.1. Length, width. Length and width I have given reciprocally:
2. A surface have I built (abni, from banûm)
3. Next, so much as length over width
4. goes beyond,
5. to inside the surface I have appended:
6. 3`3`. Again, length and width
7. I took together[11]: 27. Length, width and what surface.

27	3`3`	the accumulations
15	length	3` the surface
12	width	

8. You, by your procedure
9. 27 the accumulation of length and width
10. to inside [3`3`] append:
11. 3`30`. 2 to 27 append:
12. 29. Half of that which is 29 you break off[12]:
13. 14`30´ times 14`30´, 3`30`15´.

14. from inside 3`30`15´
15. 3`30` you tear off:
16. 15´ the remaining thing, 30´ the side.
17. 30´ to the first 14`30´
18. append: 15 the length.
19. 30´ from the second 14`30´
20. you cut off: 14 the width.
21. 2 which to 27 you appended
22. from 14 the width you tear off:
23. 12 the true (gi-na) width
24. 15 length, 12 width give reciprocally:
25. 15 times 12, 3` the surface.
26. 15 length over 12 width
27. what goes beyond?
28. 3 goes beyond, 3 to inside 3` the surface append,
29. 3`3` the surface.

[Problem 2]

30. Length, width. Length and width
31. I have given reciprocally: A surface have I built.
32. Next, the half of the length
33. and the third of the side
34. to inside my surface
35. I have appended: 15
36. Again, length and width
37. I have taken together: 7.

II.1. Length and width what?
2. You, by your procedure:
3. 2, inscription of the half[13]
4. and 3, inscription of the
5. third, you inscribe:
6. The reciprocal of 2, 30´ you find out:
7. 30´ times 7, 3`30´ to [i.e. as a replacement of] 7
8. the accumulation of length and width
9. you bring to place:
10. 3`30´ from 15 my accumulation
11. cut off:
12. 11`30´ the remaining thing
13. No further it go! 2 and 3 I give reciprocally.
14. 3 times 2, 6.
15. The reciprocal of 6, 10´ it gives to you.

16. 10′ from 7 your accumulation
17. of length and width I tear off:
18. 6˚50′ the remaining thing.
19. Half of that which is 6˚50′ you break off:
20. 3˚25′ it gives to you.
21. 3˚25′ until two times
22. you inscribe: 3˚25′ times 3˚25′,
23. 11˚40′25″. From inside [of it]
24. 11˚30′ I tear off:
25. 10′25″ the remaining thing. <It side 25′.>
26. To the first 3˚25′
27. 25′ you append, 3˚50′,
28. and that which from the accumulation
29. of length and width I tear off,
30. to 3˚50′ you append:
31. 4 the length. From the second 3˚25′
32. 25′ I tear off, 3 the width

7 the accumulation(s)
4 the length
3 the width 12 the surface

[Problem 3]

33. Length, width. Length and width
34. I have given reciprocally:
35. a surface have I built.

III.1 Next, so much as length over width
2. goes beyond, with the accumulation
3. of my length and width I have given reciprocally:
4. to inside my surface
5. I have appended:
6. 1``13`20˚. Again, length and width
7. I have taken together: 1`40˚.

1`40˚ | 1``13`20˚ the accumulations
1` the length
40 the width | 40` the surface

8. You, by your procedure
9. 1`40˚ the accumulation of length and width;
10. 1`40˚ times 1`40˚, 2``26`40˚.
11. From 2``26`40˚ 1``13`20˚ the surface
12. you tear off: 1`` 33`20˚.
13. No further it go! Half of that which is 1`40˚

14. you break off: 50 times 50,
15. 41`40˚ to 1`` 33`20˚ you append,
16. 2`` 15`,  1`30˚ the side.
17. 1`40˚ over 1`30˚ what goes beyond?
18. 10 goes beyond, 10 to 50 append,
19. 1` the length. 10 from 50 cut off:
20. 40 the width.

We shall follow this geometrically. In the first problem (see Figure 9A), to a rectangle is appended the difference between the length and the width - an addition which will only make geometrical sense if as usual a rectangle, e.g. of length x-y and width 1, is appended. The resulting bold-line surface is completed as a rectangle by the addition of x+y, a given number (again in the form of rectangles). Of this completed rectangle, the width is y+2, and thus both the sum of its length and width {(x+y)+2} and its surface {xy+(x-y)+(x+y)} are known[14]. So, we are now confronted with one of the standard-problems of Babylonian algebra (one which, accidentally, we have not yet discussed). The geometrical solution of this standard-problem is shown in Figure 9B (where, for the sake of clarity, the proportions of the figure have been changed): The sum of length and width is bisected. The rectangle is "broken" correspondingly and rearranged so as to form a gnomon of known side. So, the difference between the area of the rectangle and that of the completed square is known, and thus the side d of the difference.

d is appended to the first side (horizontal on our figure) of the completed square, which then becomes the length of the rectangle, and it is torn off from the second side, which becomes "the width", i.e. the width of the greater rectangle of which the surface and the sum of length and width were known. A further subtraction of 2 gives the "true width", i.e. the width of the original rectangle. Finally, two "surfaces" are calculated, first the surface of the original rectangle ($x \cdot y$), then the sum of this surface and the difference between length and width ($x \cdot y + x - y$).

Terminologically, this is very interesting. If instead of a geometrical we apply a rhetorical interpretation of the procedure, we have to do with a case of "change of variable", calculation in terms of x and Y, $Y=y+2$. That Y is called "the width" and y "the true width" is no argument against such an interpretation since, if the standard terms for the variables were "length" and "width", these would also have to be used in the problem $x+Y=29$, $x \cdot Y=3`30$. On the other hand, the double use (without a distinction between "true" and "false" or "preliminary") of the term "surface" makes good sense in the geometrical interpretation where both quantities are in fact surfaces

(cf. also the use of the term as a designation for xy, for x^2 and for $9 \cdot (x-y)^2$ in VAT 8390, problem 1 - see pp. 16f and 19). In a non-geometrical interpretation, however, this usage is hardly

to explain[15].

We notice that the geometrical interpretation gives a very concrete meaning to the change of variable, and that even the quantity d needs no interpretation as $\frac{x-y}{2}$, but can be used in its immediate visual meaning as "that quantity by which both length and width differ from half their sum".

In problem 2 again, $x \cdot y + \alpha x + \beta y = A$, $x+y=B$, and once again, one of the first-order terms of "equation 1" is eliminated by means of equation 2. This time, however, the matter is more complicated, since $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$.

At first, $\alpha x + \beta y$ is appended to the surface $x \cdot y$, e.g. as in Figure 10A. 2 and 3 for $\frac{1}{2}$ and $\frac{1}{3}$ are "inscribed", and the sum of length and width is multiplied (arithmetically) by ½. The result, $2(x+y)$, is brought to/in the place of x+y", and since this is not for later use in the procedure, it must be for what is immediately done. What is done is perhaps best given a meaning by Figure 10C: $\frac{1}{2}x+\frac{1}{3}y$ is shown along and inside a rectangle $(x+y) \cdot 1$, and we see that if instead of this rectangle we take another one $(x+y) \cdot ½$, the subtraction of the latter will eliminate the surface ½x, leaving us with $(\frac{1}{2}-\frac{1}{3})y$ to be subtracted from the rectangle. So, we will be back in a situation parallel to that of the first problem, cf. Figure 10B. (Of course, what is done in Figure 10C can just as well be made along the edges of the rectangle in Figure 10A).

However, in order to know which multiple of the width is subtracted from the square in Figure 10B, the scribe inserts the break "no further goes", and makes a curiously formulated calculation. $\frac{1}{2}-\frac{1}{3}$ is calculated without intermediary steps as $\frac{1}{2\cdot 3}$, which, if it is done arithmetically, presupposes that 3-2=1. This presupposition is of course not astonishing at all. More noteworthy is the fact that the multiplication is formulated twice: First as "giving 2 and 3 reciprocally", an expression which in other cases is used in connection with "rectangular multiplications" (where two entities which at least formally are lengths are multiplied to each other); then, in the next line, by the arithmetical term found in the multiplication tables, "3 times 2". Such a double formulation of the same step in a procedure is most unusual, and one could suppose that therefore the two lines do in fact tell different things. Figure 10D is a possible interpretation: Taking reciprocal giving in the usual sense, a rectangle 2·3 is made. This is a figure of which both $\frac{1}{2}$ and $\frac{1}{3}$ is easily taken. In total, it contains <u>2 times 3 small</u> rectangles. Half of it is delimited by a bold line, and the third by a barred line. The difference is <u>seen</u>, without any argument, to be 1 square out of 6, i.e. a sixth part, 10′. This could have been drawn in the corner of Figure 10A, between the half and the third; the "inscriptions" of 2 "of the half" and 3 "of the third" - precisely the numbers which are "given reciprocally" - might be taken as an indication that this is precisely what takes place.

Addendum: In TMS XVI A, obv.8-9 (cf. below, p. 87), 1-¼ is calculated not as 1-15′ but insteas by the subtraction of 1 from 4 and next by multiplication by 15′. If we interpret this not as generalized calculation with fractions (which is not to be expected) but in the style of a "false position" (the <u>mokhraj</u>, Arab term for the entity of which the whole and the ¼ is taken, being 4), and try to make the analogous consideration in this case, we will have to look for an entity of which ½ as well as ⅓ is easily taken - and a visually simply manner to demonstrate that you do this by multiplying is to make the 2·3-rectangle.

On the <u>mokhraj</u>, cf. L. Rodet 1881:205-211.

Further Babylonian texts using the method are VAT 7532 and VAT 7535, where the calculator comes e.g. from $1-\frac{1}{3}$ of a quantity to the whole by a representation of the whole by the number 5.

This avoids us two further troubles, which, truly, taken by themselves are minor, but which still add to the doubt arising from the repeted prescription of the multiplication and the unusual use of the term for rectangular multiplication: First, if a pure, unargued arithmetical calculation is meant, why then doesn't the scribe simply calculate $\frac{1}{2}-\frac{1}{3}=$ 30′-20′=10′? Next, why is the order of the factors reversed between two statements of one and the same multiplication?

This is no <u>proof</u> that geometrical heuristic was used even for certain direct arithmetical calculations. But the agreement between the formulation and a geometrical procedure, and the accumulation of smaller and greater difficulties when an arithmetical interpretation is applied, make it a noteworthy hypothesis (further support for the interpretation is given on p. 83).

From this point onwards, everything is strictly parallel to problem 1. Only two things deserve a remark: For one thing, the "length" of the shortened rectangle is not spoken of as a length, and accordingly the original length is not called a "true length". Once again we have evidence that the mathematical language was no strict technical language but only a (somewhat standardized) description of procedures (cf. p. 19). Further, we notice that in this problem, the width is eliminated, while in problem 1 this was done to the length. Had we subtracted not $\frac{1}{2}(x+y)$ but only $\frac{1}{3}(x+y)$, we would have reached a situation much closer to that of problem 1 (elimination of the width, addition instead of sub-

We notice that II.13-14 is not the only occurrence of the double construction. It is also found in I.24-25. So, there seems to be a distinction between the process of <u>constructing</u> a rectangle, the "reciprocal giving", and the calculation of the area (cf. also the alternation between "reciprocal giving" and "raising" in connexion with rectangular areas in VAT 8390 (above, pp. 16ff) and the discussion below, p. 83. In most texts, the construction is taken to imply the calculation; in the present text, both are spelled out explicitly in two cases, and in certain other places (I, 13; III, 10) it is instead the construction which is tacitly implied.

In YBC 4662, obv. 7, 18 and 30f, an explicit double construction is also found - here involving UR.UR and a-rá - see MCT, 69-72. Even UR.UR is a "rectangular multiplication" (possibly a homophonic writing of UL.UL).

traction of a number of lengths). So, the procedure is not one aiming at the creation of a fully standardized problem which can be solved by a mechanical algorithm. It is one of applying a set of flexible standard techniques[16]. (cf. the discussion at p. 26f).

The third problem might have been treated along exactly the same lines, but it is not. Symbolically expressed, x+y=1`40, x·y+(x+y)·(x-y)=1`` 13`20`. If we insert the first equation, the second becomes x·y+1`40·(x-y)=1`` 13`20`. Here, y is easily eliminated by addition of 1`40·(x+y).

However, this scheme from no. 1 and 2 is not followed here. Instead, we may try to follow geometrically the exact prescriptions of the text (Figure 11A). Then, first, from length and width a rectangular surface is made, to which is appended a rectangle spanned by the excess of the length over the width (most naturally drawn in prolongation of the width) and the sum of the length and the width. This is the bold-line surface given as 1`` 13`20`. It is subtracted from the square on the accumulation of length and width; at this point, then, we meet the "no further goes"-clause, indicating that a leap is undertaken in the argumentation.

In fact, a leap there is. The remainder (dotted lines) is seen to be one surface of the square on the width, and 1`40` sides of the same square.

This is exactly what is utilized in the solution, which follows the standard scheme for such problems: Breaking-off the half, squaring, appending, finding the side (see Figure 11B).

Now, instead of subtracting from this side 50, which would leave the width of the original rectangle, the quantity $d = \frac{x-y}{2}$ is found, and then from $\frac{x+y}{2}$ length and width. Once again, the geometrical meanings of these quantities are manifest on the figures, and no abstract algebraical skills are needed to argue for the correctness of the solution.

Before leaving these texts, we shall return to the phrase "half of that which is X" common to all of them (cf. note 12). We notice that each time it occurs, X is a quantity which is bisected in order to allow the re-arrangement of a rectangle as a gnomon. To consider the expression as a mere stylistic peculiarity of this tablet is unsatisfactory. The text uses no stylistically analogous expressions like "to that which is X, append Y", but the direct "to X append Y", nor any extended genitives. So, "that" which is X must be considered <u>something which has an existence beyond that of being X</u>; something, furthermore, which is not easily designated verbally. In the first two cases X represents the sum of length and width of the modified rectangle, which, should it be verbally described, would require a rather heavy phrase. In the third case, X is in reality "the accumulation of length and width" (and is so designated in line III.9). If this is no adequate description in line III.13, it seems a necessary implication that it is at this place no longer the relevant description[17]; if instead it should be thought of as "the length of the lower dotted rectangle of Figure 11A", as a "that" accompanied by a pointing gesture or thought, the phrase becomes meaningful.

Other variations of the second-degree-theme are provided by the tablet YBC 6504 (MKT III, 22f). Since little new information could be extracted from a detailed textual analysis, I shall renounce the line-by-line translation.

All four problems deal with a rectangle, from which the square on the excess of the length over the width is torn off, $x\cdot y-(x-y)^2=8`20'$. The variation is provided by the variation of the second statement:

Problem 1: The excess of length over width, $x-y=10$.

Problem 2: Length and width taken together, $x+y=50$.

Problem 3: The length, $x=30$. "Its width" is asked for.

Problem 4: The width, $y=20$. "Its length" is asked for

In all cases, the length of the rectangle is 30 and the width is 20. So, the concern of the tablet is the systematic exposition of the variation of methods according to the possible combinations of "givens", a concern which is shared by many Babylonian mathematical texts.

Symbolically expressed, the four problems are solved as follows:

Problem 1

$$xy - (x-y)^2 = 8`20', \quad x - y = 10 .$$
$$xy = 8`20' + 10^2 = 10`$$
$$xy + (\tfrac{x-y}{2})^2 = (\tfrac{x+y}{2})^2 = 10` + 5^2 = 10`25'$$
$$\tfrac{x+y}{2} = \sqrt{10`25'} = 25$$
$$x = 25 + 5 = 30 \qquad y = 25 - 5 = 20$$

Problem 2

$$xy - (x-y)^2 = 8`20', \quad x + y = 50 .$$
$$xy - (x-y)^2 + (x+y)^2 = 8`20' + 41`40' = 50` , \text{ or}$$
$$xy = \tfrac{1}{5} \cdot 50` = 10`$$
$$(\tfrac{x+y}{2})^2 - xy = (\tfrac{x-y}{2})^2 = 10`25' - 10` = 25$$
$$\tfrac{x-y}{2} = \sqrt{25} = 5$$
$$x = 25 + 5 = 30 \qquad y = 25 - 5 = 20$$

Problem 3

$$xy - (x-y)^2 = 8`20', \quad x = 30 .$$
$$x^2 - xy + (x-y)^2 = 15` - 8`20' = 6`40' , \text{ or}$$
$$(x-y)^2 + 30\cdot(x-y) = 6`40'$$
$$\{(x-y) + 15\}^2 = 6`40' + 3`45' = 10`25'$$
$$(x-y) + 15 = \sqrt{10`25'} = 25$$
$$x - y = 25 - 15 = 10$$
$$y = x - (x-y) = 30 - 10 = 20$$

Problem 4

$xy - (x-y)^2 = 8`20^{\cdot}, \quad y = 20$

$xy - (x-y)^2 + y^2 = 8`20^{\cdot} + 6`40^{\cdot} = 15`$

This equation, which could be reduced to

$3xy - x^2 = 15`$ or $(3y-x)\cdot x = 15`$,

is interpreted as

$x^2 = 15`$, from which then follows

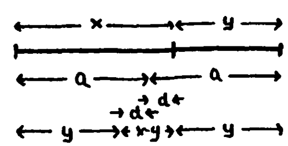
$x = \sqrt{15`} = 30$.

The formulations are most unkind to the reader. Inside the calculations, none of the numbers are accompanied by explanations of their sense, nor even of indications as to their origin if they are results of earlier calculations. Everything is a statement of the algorithm, in the style of BM 34568, problem 9 (see p. 3).

The geometrical interpretations of the four problems are shown in Figures 12-15. In the first three figures, the proportions of the rectangle are changed in order to avoid misunderstandings due to the accidental identity between x-y and ½y. In the fourth problem, exactly this accidental equality appears to be the root of a short-circuit in the argument, and so the figure has to keep the proportions if this is to be understood.

In problem 1 (x-y=10; cf. Figure 12), the squared excess $(x-y)^2$ is appended to the incomplete rectangle $x\cdot y-(x-y)^2$. We are then brought

back to the standard situation, a rectangle of which the surface and the excess of length over the width are known, and the further procedure runs as normally (cf. Figure 12B). This is what could be expected, if geometrical reasoning was used. In the opposite case, however, the procedure constitutes a detour. In fact, if we translate the standard-method into symbolic calculations, it is based on the equality $x\cdot y = (\frac{x+y}{2})^2 - (\frac{x-y}{2})^2$, from which, when $x\cdot y$ together with $\frac{x\pm y}{2}$ is known, $\frac{x\mp y}{2}$ is calculated. If we interprete the solution of the complete problem along these lines, first x-y is squared and added to $x\cdot y-(x-y)^2$; then x-y is bisected, and its half is squared and added. So, when neither $(\frac{x-y}{2})^2$ is calculated directly as $(x-y)^2 \div 4$ nor the two additions are combined into one addition of $5\cdot(\frac{x-y}{2})^2$, this must be considered still further evidence against the presence of an abstract, non-geometric understanding of the algebraic transformations in question.



If the recurrent way to split the sum of length and width, viz. by means of halfsums (a) and deviations (d), is taken into account, the dissection of Figure 13A is probably the most natural splitting of the square on the sum of length and width. At the same time it shows immediately that the square equals 4 rectangles plus that square on the excess which was cut off from our fifth, original, rectangle.

In problem 2 (x+y=50), the sum of length and width is squared and appended to the incomplete rectangle. One possible way to do this together with one possible dissection of the square is shown in Figure 13A. The sum is multiplied (arithmetically) by 5^{-1} and, in fact, we notice on the figure that the shaded rectangle subtracted from our original rectangle is equal to the central, dotted square of the dissection. If this is taken into account, the surface of the sum can be seen as 5 times the surface of our original rectangle,- as inherent in the text.

After the multiplication by 5^{-1}, the text proceeds as usual for a rectangle where the surface and the accumulation of length and width are known (Figure 12B).

Once again, this is the natural way to proceed for an algebra based on concretely seen geometric heuristics and with the "accounting technique" (see p. 27) at its disposal (even the arithmetical division via a multiplication by 5^{-1}=12′ is nothing but a natural consequence of the accounting). But if an abstract, non-geometric understanding of the matter was present, the procedure constitutes another detour. In fact, in the first part of the calculation, $(x+y)^2$ is calculated and added and the sum divided by 5. In the second part, x+y is bisected, $(\frac{x+y}{2})^2$ is found, and the result of the division by 5 is subtracted. For the abstract, non-geometric way of thought used to express itself in the quantities $\frac{x+y}{2}$ and $\frac{x-y}{2}$ (and these were, abstractly or concretely, basic entities in Babylonian algebra), the defect rectangle could be expressed as $(\frac{x+y}{2})^2-(\frac{x-y}{2})^2-(x-y)^2 = (\frac{x+y}{2})^2-5\cdot(\frac{x-y}{2})^2$ = 8`20', and if this was known, the obvious solution would be $(\frac{x-y}{2})^2 = 5^{-1}\cdot\{(\frac{x+y}{2})^2-8`20'\}$.

In passing we may notice that the dissected square of Figure 13A is transformed into a

well-known heuristic proof of the Pythagorean theorem (Figure 13C) just by the drawing of diagonals in the rectangles.

In problem 3 (x=30), the length is squared and the incomplete rectangle is subtracted. We see immediately on the figure, that the remainder consists of the square on the excess of length over width (heavy shading) and 30 times this same excess. In fact, the solution proceeds as a normal "surface plus 30 sides"-problem, finding the excess (without mentioning its name) to be 10 and thus the width to be 30-10=20.

This problem has always been seen to represent a "change of variable" (see MKT III:25, and TMB p. xxv). According to the assumption of a geometrical heuristic, the idea of a "change" is somewhat misleading. Instead, the adequate "variable", i.e. the adequate quantity to submit to the usual treatment, is chosen among the entities present on the figure according to convenience - and in this case, the excess is certainly the most convenient choice. The "changes" of width and length in AO 8862, problem 1 and 2, respectively, should be understood according to the same principle, although, in their case, the usual term "width" is used even for the new width. In fact, the new "width" is a width much more truly than, in symbolic algebra, Y is "a y".

In problem 4 (y=20), all that is explicitly formulated in the text is the addition of the square of the width, the result of which should, according to the text, be the square of the length. It is, indeed, but only due to the relation 3y-x=x in this specific case.

The error is easily explained if we suppose a geometric heuristic making use of a figure with the correct proportions. In Figure 15A the defective rectangle is shown, and in Figure 15B it it rearranged so as to make sense of the addition of y^2. In Figure 15C y^2 is added, and the result is "seen" to be a square of side x.

If, on the other hand, we try the same procedure on a rectangle of different proportions (Figure 15D-F), we get a rectangle of dimensions $x \cdot \{y+(y-(x-y))\} = x \cdot (3y-x)$.

Neugebauer (MKT III, 25) conjectures a way in which the error could have arisen by means of a procedure parallel to symbolic algebra, and Gandz (referred in TMB, p. xxv) extends this into an explanation how abridged formulations of a correct procedure could explain the formulations of the text as fundamentally correct. In principle, the point of the two explanations is a combination of the two basic ~~equations of the problem into one,~~ $1` \cdot x - x^2 = 8`40` + 20^2 = 15`$. This can, of course, also be achieved by means of geometric heu-

Even Bruins (1971:251) suggests an abridged procedure.

ristics, albeit by means of a considerable amount of book-keeping. "Equations" of this type ("sides minus square equals number") are never found explicitly in the Babylonian texts, but in a certain number of cases they seem present behind the curtain (one instance is BM 85194, Rev. II.7ff, see below, p. 57). Now, if equations $2a \cdot x - x^2 = b$ are solved by means of geometrical heuristics, fundamentally different figures have to be used for the two cases $x<a$ and $x>a$ (i.e., in order to find the double solution one has to make use of several figures) - see figure 16. This would of course give rise to only limited troubles for the Babylonians, who normally knew the solution to their artificial second-degree-problems in advance. But in any case, if like here the solution is known to be $x=a$, the Babylonians would also have known to be in the limiting case where the gnomon to be transformed is already a square, and the abbreviated formulation would be justified.

Cf. also Thureau-Dangin 1940:9f.

So, even under the hypothesis of a geometrical heuristic we cannot exclude Gandz's explanation of the procedure as correct. On the other hand, the complicated book-keeping required does not fit the style of the elegant solutions to problems 1-3, and so the error seems to me to be the more plausible assumption.

Terminologically, one contribution of the tablet is of a certain interest: The term __šakanum__, "to pose [for action upon]", *Another terminological point can be taken note of: In the first two problems, the term íb-si₈, elsewhere used to express the "side of square" (arithmetically seen the square-root) is used in the enunciation of the problem for the square on the excess which is to be torn off. In the description of the solution, the term šutamḫurum, "to raise against it equal", is used when squares are made. In problems 3-4, the term UL.UL (written ZUR.ZUR in MKT) is used in both functions. In all four, íb-si₈ is used when square-roots are extracted.*

It seems that íb-si₈ is used to speak of a square figure which is already there - the term is a static description, cf. also the etymology from si₈, "be equal", and note 9a. Šutamḫurum describes (or prescribes) a construction, a procedure, as is to be expected from the grammatical form. UL.UL is, so it seems, not used logographically for any of the two but as a more-embracing ideogram - cf. also the addendum p. 64f.

the possible connotations of which in the nature of "drawing" or "inscribing" were already mentioned (see n. 10 and p. 29) appears in this text (or rather, its Sumerian substitute in-gar appears) not only when numbers are taken notice of which will be used or acted upon later in the solution of a problem, but even when the final results are noted. So, in this text, the sense of "inscribing" is obvious.

We shall close the treatment of second-degree problems by a few texts which are not explicitly formulated as problems concerned with the plane figures which enter into the heuristics, but which still use the same techniques.

First we shall look at problem 12 from BM 13901 (MKT III,3). Superficially regarded, is looks exactly like the other problems of that tablet (see p. 12f and 20f)[18]:

Obv.II.27. The surfaces of my two square figures I have taken together: 21`40°.

28. (The sides of) my two square figures I have given reciprocally: 10`.

29. Half of 21`40° you break off: 10`50° and 10`50° you give reciprocally,

30. 1```57`` 21`40°. 10` and 10` you give reciprocally, 1```40``

31. inside 1```57`` 21`40° you tear off: 17`` 21`40°; 4`10° the side.

32. 4`10° to the first 10`50° you append: 15`, 30 the side.

33. 30 the first square figure.

34. 4`10° inside the second 10`50° you tear off, 6`40°, 20 the side.

35. 20 the second square figure.

Symbolically, what happens is this:

$x^2 + y^2$ = 21`40° , $xy$ = 10` .

$\left(\frac{x^2+y^2}{2}\right)^2$ = 10`50°^{2} = 1```57`` 21`40°

$\left(\frac{x^2-y^2}{2}\right)^2 = \left(\frac{x^2+y^2}{2}\right)^2 - (xy)^2$ = 1```57`` 21`40° - 1```40``
= 17`` 21`40°

$\frac{x^2-y^2}{2}$ = √(17`` 21`40°) = 4`10°

$x^2 = \frac{x^2+y^2}{2} + \frac{x^2-y^2}{2}$ = 15`, $x$ = √15` = 30

$y^2 = \frac{x^2+y^2}{2} + \frac{x^2-y^2}{2}$ = 6`40°, $y$ = √6`40° = 20

This is easily followed geometrically on e.g. Figure 9B, as it follows the standard procedure for rectangles for which the surface and the accumulation of length and width are known - provided that x^2 and y^2 are considered as length and width of a rectangle, and $(xy)^2$ as its surface.

It will be observed that the squaring of xy, i.e. the finding of the "surface" spanned by the two surfaces x^2 and y^2, is expressed as "reciprocal giving". As far as dimenionality is at all expressed by the Babylonian terminology (which is only partly the case), the term chosen for the multiplication is "correct". In other words: A "surface" spanned by "surfaces" is still regarded a "surface", not just an arithmetical product, as long as we move inside the representation.

So, the concentration on concrete thought and geometrical heuristics should not make us believe that Babylonian algebra was nothing but a technique for the calculation of concrete figures by cutting, pasting and counting. Obiously, that level of abstraction which allowed the identification of any sum or difference with the sum or difference of the basic entities "length" and "width" was present. The concreteness of methods was apparently no obstacle to the generality of their application. What seems concrete to us was used as a fully abstract means of representation (a fact which may have furthered the eventual transformation of geometrical into arithmetical thought).

Exactly such a use of the concrete as a means to represent what the modern mind would think of as general or abstract classes is discussed amply by Levi-Strauss in The Savage Mind (1972). He characterizes (pp. 16ff) the "savage" philosopher as a bricoleur with regard to concepts,- the French term bricoleur expressing the idea of a non-professional and non-specialized artisan of all trades who works with the means accidentally at his disposal and not with specialized tools, fabricating when needed a water-pump from the remains of a grand-father-clock and an old car. In this sense, the Babylonian is still related to the bricoleur, and Babylonian mathematics is still tainted by "savagery".

But only related, and only tainted. If we look at the text just discussed, or at the commercial problem where two prices were represented as the length and the width of a rectangle (p. 7), we notice that the entities seemingly treated as geometrical sides are not designated as such - we are already halfway between "savage" concreteness and that Euclidean abstraction which allows abstract magnitudes to be represented in the demonstrations of Elements V by lines.

Other problems making use of the second-degree techniques do not look at all like the standard problems of squares and ~~"length and width". One instance of this is~~ a set of strictly parallel problems of VAT 7532 and VAT 7535. We may look at the version of VAT 7532 (MKT I,294f).

A trapezoid (see Figure 17) is measured by means of a reed of unknown length R. At the (upper) length (drawn on the tablet as perpendicular to the parallel widths), 1` steps are made with the reed, then the sixth part of the reed breaks off, and further 1`12' steps are made.

Of this new reed, one third and a third of a cubit then breaks off, and the "upper" (left) width is gone through in 3` steps. The second broken-off piece is restituted, and the "lower" (right) width is measured in 36 steps. The area is 1 bur, i.e. 30` measured in squares of GAR, the basic unit of length, of which the cubit is 1/12. The original reed R ("head of the reed") is asked for.

In order to follow the procedure, we shall call the length of the reed once shortened r, and that of the reed twice shortened z. Then, the length is 1`·R+1`12'·r, the upper width is 3`·z, and the lower width is 36·r.

$r = \frac{5}{6}R = 50'\cdot R$, $R = 1;12'\cdot r$, $z = \frac{2}{3}r - \frac{1}{3\cdot 12}$
$= 40'\cdot r - \frac{1}{3\cdot 12}$.

What goes on seems to be the following[19]:

Everything is calculated in terms of r (which, as we have discussed above, p. 26 and note 10, is posed as "1"). The length is found to be 1`·1;12'·r+1`12'·r = 2`24'·r, the upper width is 3`·z = 2`·r - 5, and the lower width of course 36·r.

Next, an auxiliary rectangle is considered[20], composed from two trapezoids with length 2`24'·r, upper width 2`·r and lower width 36·r. Each

The terminology used supports the idea of a real doubling of the figure. The area of the original trapezoid is not "raised to 2", it is "doubled until 2 times" (tab~ eṣēpum), i.e., it is repeated as a real entity, not replaced by its measuring number.

of these two trapezoids exceeds the original trapezoid by a triangle with height 2`24˚·r and base 5. So, the area of the auxiliary rectangle exceeds 2·30`=1``, the double area of the original trapezoid, by the area of the broken-line parallellogram, i.e. by 2`24˚·5·r = 12`·r. Its own area is (2`24˚·r)·(2`·r+36·r) = 2`24˚·2`36˚·r = 6`` 14`24˚·r^2 (expressed in the text simply as 6`` 14`24˚, since r is posed as "1"). So, we have 6`` 14`24˚ surfaces of the square on r diminished by 12` sides - a second-degree standard-problem which is solved precisely as one would expect it to be solved. Finally, the original length of the reed is calculated from r as $r+\frac{1}{5}r$.

We shall close the discussion of second-degree techniques by a group of closely related problems which show the explanatory power and, as far as one of them is concerned, the limitations of the concepts of a geometric heuristic developed in the above.

All problems deal with a siege ramp which is at the moment of the calculation not yet completed (see Figure 18A and, for the real proportions of the construction, Figure 18C). It is planned to reach the city-wall at height h and to have a length ℓ, but only a length ℓ_1 and a height h_1 has been built, while a final length of Δ is still lacking. (As always, the horizontal dimensions are silently presupposed to be measured in GAR, the height in cubits and

volumes correspondingly in GAR·GAR·cubits - 1 cubit = 5′ GAR). In those three of the problems which I discuss (the fourth is too corrupt to permit any secure interpretation) the width of the construction is given to be 6 GAR and the total volume of earth required to be 1``30`, from which in all cases the total surface F of the plane section of Figure 18A is calculated to be 15`[GAR·cubits].

We begin by the problem BM 85194, Rev.II.22-33. Here, apart the information implying F=15`, ℓ_1=32 and h_1=36 are given. Δ is asked for.

h is calculated from

$$h^2 = (2F)\cdot\frac{h_1}{\ell_1} = 33`45˚, \text{ whence } h = 45.$$

This leads to $h-h_1$ = 45-36 = 9, and so

$$\Delta = (h-h_1)\cdot\left(\frac{h_1}{\ell_1}\right)^{-1} = 8.$$

This is fully correct, and as Neugebauer has shown (MKT I, 183f), these formulas can be derived by a combination of the formula for the surface of the triangle and the proportion

$$\frac{h}{\ell} = \frac{h_1}{\ell_1}.$$

So, everything should be in order. However, the route followed by our text is not the obvious one if such techniques were used. Quite as easily found as the formula for h^2 would be

$$\ell^2 = (2f)\cdot\frac{\ell_1}{h_1},$$

which would spare us one multiplication by a three-place sexagesimal and one inversion of the same number. We must therefore suppose that the problem was not solved directly by arguments of proportional triangles but rather according to some scheme common to all problems concerned with this construction.

Such a scheme is easily provided by the geometrical heuristics as we have come to know it. If first we double the triangle of surface F, as the trapezoid was doubled in VAT 7532 (Figure 17), we get a rectangle of height h and width ℓ (dotted line in Figure 18B).

Once again, indeed, the area of 15` is "doubled", i.e. the triangle repeated.

By a change of horizontal scale analogous to that used in the solution of equations $a \cdot x^2 + b \cdot x = c$ (see Figure 4 A-B), this can be transformed into a square. The factor of scale can be expressed at will and according to calculational convenience as $\frac{h}{\ell}$, $\frac{h_1}{\ell_1}$ or $\frac{h-h_1}{\Delta}$, and the surface of the square will be h^2. At the same time, it can be expressed as 2F multiplied by the scaling factor.

It is worth noticing that the scaling factor is calculated before anything else. The first thing to come to the Babylonian mind is not, as to ours, to reduce the problem to a plane, triangular problem which can then be treated by means of proportional figures. Instead, the first thing to be looked for is that basic scaling factor which reduces to a "quadratic situation". Then, the width is eliminated, and we are left with a triangular area which is first "raised" to the scaling factor (which gives us the area of the corresponding right, isosceles triangle) and then "doubled" (which gives us the area of a square on h).

This will at once give us the fundamental formula of the above solution,

$$h^2 = (2F) \cdot \frac{h_1}{\ell_1} .$$

Of course, a change of vertical scale could just as easily have been made, from which would have followed a formula for ℓ^2. So, the explanation by geometrical heuristics is only an improvement on Neugebauer's justification of the calculation (no claim is advanced by Neugebauer that the Babylonians followed his calculations) if the transformation of Figure 18B was a standard procedure not invented for special application in this problem.

The application of the procedure as a general tool is supported by BM 85210, Obv. II.15-27. Once again, apart volume and width (and thus the surface F = 15`), $h_1$ = 36 and $\ell_1$ = 32 are given. h and ℓ are asked for. This time, however, the surface $F_1$ = 9`36˙ (corresponding to the already completed part of the ramp) is found, whence F_2 = 5`24˙. h is found from

$$h^2 = (2\frac{h_1}{\ell_1}) \cdot F_2 + h_1^2$$

and ℓ finally as $\frac{F}{\frac{1}{2}h}$.

This correct but seemingly strange procedure is made understandable by the transformation of Figure 18B. We notice that the surface of the square is composed from a square equal to h_1^2 and a gnomon, and that this gnomon is composed from two equal trapezoids. One of these is seen immediately to be equal to F_2 multiplied by the scaling factor. So, the total area of the gnomon is $(2\frac{h_1}{\ell_1}) \cdot F_2$, and the above formula for h^2 is found.

Thus, this second solution to the same problem is explained by the assumption of a common heuristic figure which were known to both authors. However, if we assume that not the geometrical figure or procedure was known but the corresponding formula for h^2, the calculation of this second problem is an enigma or, at least, an anomaly. The **assumption of a shared heuristic seems better supported that the assumption of a** shared stock of formulae (that something was shared is evident from the complete coincidence of all dimensions of the ramps).

The third problem concerning the ramp is more complicated. It is found in BM 85194 Rev.II.7-21 (immediately before the problem discussed here as the first one). Given are as usually the width of the ramp and the total volume of earth, from which the surface F = 15` is calculated. Given are further $\Delta = 8$ and $h_1 = 36$, and asked for is ℓ (and ℓ_1 which however is not calculated after all). The solution seems to go via an equation

$$(2\frac{F}{\Delta})\cdot h - h^2 = (2\frac{F}{\Delta})\cdot h_1$$

which, if geometrically solved, is treated according to the procedure of Figure 16A. Alternatively, a system of two equations (a standard system of "surface and length together with width") is used,

$$h + H = \frac{2F}{\Delta} \qquad h \cdot H = \frac{2F}{\Delta} \cdot h_1$$

where the auxiliary quantity H is $\frac{h_1 \ell}{\Delta}$.

Two questions then arise. Why was this done, and how was this done?

Neugebauer (MKT I, 183) justifies the correctness of the procedure by symbolic transformations of the proportion $\frac{\ell}{\Delta} = \frac{h}{h-h_1}$ and the equation $2F = h \cdot \ell$, and Goetsch (1968:138f) postulates this to be the real procedure. Two difficulties, however, arise. Firstly, why wasn't an equation for ℓ,

the quantity asked for, made directly? It follows from exactly the same equations as those used by Neugebauer that

$$\frac{2F}{h_1}\ell - \ell^2 = \frac{2F}{h_1}\Delta .$$

Of course, the formulations of questions stating the required result are sometimes sloppy - so even in this problem, where ℓ_1 is asked for but not found. But the consistent sloppiness of two consecutive problems, forgetting in both cases to ask for h, is not very plausible. The detour via h gives rise to a considerable amount of extra calculation (a fact which can be foreseen in advance), and so it must be supposed, as in the first problem, to have a reason.

The second difficulty concerns the ability of the Babylonians to perform the mental operations required. The proportion in question is one involving three unknown quantities (ℓ, h and $h-h_1$), and so, any manipulation leading forwards to the equation in h must presumable involve ratios with an unknown quantity in the denominator, or other operations almost as difficult to formulate. Even if ratios and transformations of scale (with numerically given scaling factors) are attested in other texts as they seem to be, the presuppositions of such abstract manipulations would constitute something new.

Exactly the latter difficulty made Vogel propose (1936:710) an alternative. Like the first step of the transformation leading to

Figure 18B, he doubles the triangle and makes a rectangle (see Figure 19). According to Elements I, 43, the two rectangles HE and BE are equal. This could, according to our interpretation of the previous problem, easily be known by the Babylonians: when a square is made from the rectangle, the two rectangles become equal; however, their areas are changed by the same factor of scale. Then the total area of the rectangle is the sum of $\ell \cdot h_1$ (the area of AD) and $\Delta \cdot h$ (the area of HE, since it is equal to BE). At the same time, it can also be calculated as $\ell \cdot h$. So,

$$\ell \cdot h = 2F \qquad h_1 \cdot \ell + \Delta \cdot h = 2F \quad \text{or}$$

$$h + H = \frac{2F}{\Delta} \qquad h \cdot H = \frac{2F}{\Delta} h_1 \quad \left(H = \frac{h_1 \ell}{\Delta} \right).$$

Obviously, this could have been done by the Babylonians. No insurmountable cognitive hindrances seem to exist, and the method seems close to what is made in other cases. But the other trouble remains: It is just as easy to transform the two expressions for the double area into a system of equations involving ℓ as into the system leading to h. Furthermore, had Vogel's way been used, the normal habit of the Babylonians would have been to find both h and H, and afterwards to go from H to ℓ (compare AO 8862 problems 1 and 2, pp. 31-33). So, even though Vogel's proposal (made, we should remember, already

Cf. a related discussion below, p. 105.48.

before the Mathematische Keilschrift-Texte were published) is both cognitively possible and congenial, it is seems to me to be no longer quite plausible in the precise form in which it stands.

As an alternative, I would propose a return to Figure 18B, which, admittedly, can be viewed as a further elaboration of Vogel's conjecture. As it will be remembered, the total area of the square h^2 could also be expressed as 2F times the scaling factor which, for the purpose of this problem, can be expressed as $\frac{h-h_1}{\Delta}$, i.e., expressed as the Babylonians might perhaps have done, as that by which 7′30″ sides of the square exceeds 4°30′ (Δ^{-1} = 7′30″, $\Delta^{-1} \cdot h_1$ = 4°30′). So, since 2F = 30`, and since 30`·7′30″ = 3`45° and 30`.4°30′= 2``15`, the square equals that by which 3`45° sides exceeds 2``15` - symbolically expressed,

$$\frac{2F}{\Delta} h - h^2 = \frac{2F}{\Delta} h_1 .$$

So, we are lead to the required equation and, if we suppose once again that the attainment of a square was, in agreement with a fixed heuristic habit for this problem, always made by a change of horizontal scale, we are necessarily lead to the equation for h and not for ℓ.

Although the evidence for the use of a construction like Figure 18B is weak when each of the three problems is taken for itself, the coherence of the picture which it creates from the totality of related problems must be considered an important support for the conjecture. Since, furthermore, the conjecture agrees so well with the methods by which so many other problems seem at a closer investigation to have been solved[21], I shall tentatively accept it.

We notice, however, that the procedure of the third problem is not kept inside the limits of that geometric heuristic which we have discussed until now. It is not restricted to cutting, pasting and accounting: An unknown quantity is used as a scaling factor. So, although the lot of those free algebraic manipulation with unknown quantities familiar from most accounts of Babylonian mathematics can be replaced by intuitively and visually transparent geometrical procedures, a residual remains which, albeit made more transparent by geometrical representations, remains algebraic also in more modern senses of that word[22].

- be it an unknown quantity to which a concrete significance can be given in a rather simple way in terms of a line occurring in the problem ("that by which 7´30˝ sides of the square exceeds 4°30´"). This fundamental role of the scaling factor will be confirmed later in the discussion of the partition of triangle and trapezoid (in the appendix). So, the ramp problem constitutes a bridge between the normal algebra and the refined "geometric" problems.

Two final observations should be made before we leave these texts. One concerns the vocabulary. All multiplications of the three problems except the two squarings are described by the concept of "raising" (našûm). Grosso modo, this is what we should expect from the above observations on the use of this term, since most multiplications are connected to the

formation of ratios, the calculation of scaling factors, and related acts (cf. p. 15). But even in two multiplications by which an area is formed (BM 85194, Rev.II.20 and BM 85210 Obv.II.18), one factor is "raised" to the other, where we would rather expect reciprocal giving. Šutakulum designates the construction of the rectangle, the surface of which is then understood to be of course the product; našûm, on the other hand, designates directly the calculation (in this case of the surface). The use of the latter term must then be seen as a reflection of a tendency towards more arithmetical ways of thought. This tendency might represent a general trend in Babylonian algebra, since both tablets belong to the group of slightly younger modernizations of originals contemporary with those discussed above (see Goetze, in MCT p. 150f); or it might be a simple off-set from those "raising"-multiplications which dominate the calculatory interest of both tablets (of course, the two explanations might be supplementary rather than contradictory).

***Addendum:** The argument to the right is irrelevant to the matter. In both cases, areas of triangles are calculated, and this is in fact always done by "raising" (or "lifting", a semantic equivalent). The same holds true for the calculation of the areas of trapezoids and irregular quadrangles. As documented below, p. 83, "reciprocal giving" is in reality no description of a calculation but a prescription for construction; calculation, be it of areas, is covered by the term "raising", but normally only tacitly implied, not stated, when a cosntruction has been mentioned. Cf. also the marginal note p. 38.*

The present text is thus no evidence for a trend toward arithmetization of late Old Babylonian mathematical thought. Such evidence must be looked for elsewhere (cf. below, p. 105.50.

The other observation concerns the choice of units. As already explained, the unit for the vertical dimension is only one twelfth of the unit of the horizontal dimensions. Of this, the Babylonians were of course well aware, and in other (contemporary) texts dealing with dig-outs, the information that length is equal to depth is to be interpreted geometrically, not in terms of the measuring number (e.g. BM 85200+VAT6599, no. 12 - MKT I, 196). However, when the section of the ramp is trans-

***Similarly**, in YBC 4662, N° 21 (MCT, 70), the depth (z) is stated to be 1/7 of the ~~difference between length~~ (x) and width (y), while in number z=6 (cubit), and x-y = 3½ (n i n d a n).*

formed as in Figure 18B, the "square" is only a square in terms of measuring numbers, not when their geometrical meaning is taken into account - or, otherwise expressed, the figure considered for the heuristic considerations is not the real figure dealt with, it is a freely imagined auxiliary figure like the rectangle representing the two prices in Figure 3. The geometrical character of the problem and the geometrical character of the procedure are relatively independent phenomena.

SUMMING UP THE EVIDENCE

In the above, several discussions were carried through under the presupposition that the assumed geometrical heuristic was the real method of Babylonian algebra. However, due to the exposition through the investigation of single texts, the complete evidence in favour of this assumption was never set forth and weighed in its totality. This is the place to repair that defect.

The evidence is can be decomposed into three main categories: Terminological, procedural, and "cognitive".

Two sorts of terminological evidence can be distinguished: The categorical structure of the vocabulary, and the semantics of the terms.

As to the categorical structure of the vocabulary, one will notice that the classical

editions of Babylonian mathematics (MKT, TMB, etc.) translate various expressions as implying the one concept of addition, various others as implying subtraction, etc. We can list those which were met in the above (which are, certainly, the main ones):

Addition: kamārum, "to take together"; waṣābum ana / ana libbi), "to append to/to inside".

Subtraction: ina/libbi/ina libbi ...naṣāḫum, "from/inside/from inside ... to tear off"; ina ... ḫarāṣum "from ... cut off"; eli ... watārum, "over ... go beyond".

Multiplication: X a-rá Y, "X times Y"; ana N eṣēpum, "until N to double", i.e. "to multiply by N"; ana N našûm, "to raise to N"; šutākulum, in constructions with u/ana/itti/ina, "to be given reciprocally", constructed with and/to/together with/by means of"; NIGIN, "to turn [into a frame]".

Squaring: multiplicative constructions with a-rá, NIGIN and šutākalum; šutamḫurum, "to raise against its equal"; íb-si$_8$, a Sumerian term involving equality and, mathematically, presumably "the square figure" identified with its linear extension as the basic sense[23]. Cf. also the marginal note on p. 49.

Division: minam ana X šakānum, "what to pose to X [in order to get Y]", X/Y if Y is irregular; constructions with the formation of the reciprocal if Y is regular; and ana šina ḫipûm/½-šu ḫipûm, "into two to break"/ "its half to break off", if a bisection is intended.

Occasionally used as subtractive terms are: kašātum, "cut away" (from a rate of exchange in TMS XIII, cf. above p. 7, and below, p. 84); ḫaṣābum, "break away" (from a measuring cane as well as from the "1" representing its length e.g. in VAT 7532); šatāpum, "to make leave" (1, one sixth of 6, in VAT 7532). On subtractive use of paṭārum (written du$_8$), see below, p. 105.36, note (*).

In YBC 4608, obv. 24 and 27, tabālum, "to take away", is used when one member is subtracted from a sum of two pre-existing, distinct members.

***Addendum:** Another term mainly used for squaring is UL.UL. In YBC 6504, Rev. 1 and 11 it stands parallel to íb-si$_8$ as "a square figure". In Str. 363, passim, it is used for squaring, e.g. "5 UL.UL-ma 25 in-sum", "5 UL.UL: 25 is it" (Rev. 11). (Neugebauer reads the sign as zur-zur, cf. however MEA pp196-199, no. 437 and 441). The sign is also used for squaring in Str. 368, VAT 7532, VAT 7535 and VAT 7620. In Str. 363, Rev. 15-16, it is used for "rectangular multiplication", "40 ù 5 UL.UL-ma 3'20' in-sum". Cf. also p. 38 on UR.UR.*

Thureau-Dangin (TMB p. 219, "kwl") identifies the sign as an ideogram for šutākulum. However, šutākulum is never used alternating with íb-si$_8$, and only twice in

MKT it is used for squaring (except, of course, the numerous cases where x^2 is found as "x given reciprocally to x", i.e. as a product with two factors). The normal use of **UL.UL**, *on the other hand, is close to that of šutamḫurum, i.e. squaring one entity. So, šutākulum, šutamḫurum,* **UL.UL, NIGIN, LAGAB** *and* **íb-si₈** *all belong to the same broad semantic field, that of geometrical squaring and rectangular multiplication; but apart* **NIGIN, LAGAB** *and perhaps* **UL.UL**, *the terms are used in admittedly overlapping but still different ways. Neither* **UL.UL** *nor* **NIGIN** *or* **LAGAB** *are better understood if they are only seen as logograms for one of the others. Cf. also pp. 49 and 98, the marginal notes.*

These expressions are not applied indiscriminately. If, e.g., we regard additions, the term "to take together" is only used in cases where, in the geometrical interpretation, the sum of numbers can be meant (including, in BM 13901, cases where the enunciation of a problem can be meant to speak of the sums of numbers measuring a surface and a side); "appending", on the other hand, is only used where geometrical merging or some other sort of identity-conserving extension can be meant - avoiding even such cases where surfaces of squares are added but where their geometrical unification has no meaning for the procedure of solution. Similarly, in the case of multiplication, a-rá is used where one number multiplies another number, "doubling" only where a multiplication of any quantity by an integer is meant; "raising" is chosen when an operation involving proportionality, scaling and the rule of three or the calculation of areas of triangles, trapezoids and irregular quadrangles is described, "reciprocal giving" (or, in Susian and other late Old Babylonian texts, "turning") in cases where according to the geometrical interpretation a rectangular surface is "built" (banûm, cf. p. 31). "Posing" is used when a multiplication is used as a substitute for the division by an irregular number[24].

However, if the different terms were so carefully kept apart by the Babylonians, they must necessarily have designated different concepts. Addition was not one thing but two different operations, while at least

four different "multiplications" existed (which were of course known to be isomorphic to the numerical multiplication of the tablets). Such a distinction between different multiplications (or additions) is, however, not possible inside the usual arithmetico-algebraic interpretations of Babylonian mathematical thought - arithmetically, a multiplication is a multiplication; on the other hand, the geometrical interpretation creates just the conceptual distinctions required. So, if this does not prove the truthfulness of the geometrical interpretation, at least it shows that any other acceptable interpretation must to a large extent create the same distinctions as those arising from geometrical heuristics (I have not succeeded in finding anyone).

Expressed in mathematicians' argot, the arithmetico-algebraic interpretation is grosso modo homomorphic with Babylonian algebra; the geometric interpretation, on the other hand, appears to be if not identical then at least an isomorphism.

The other aspect of the terminological evidence had to do with semantics. Truly, the semantical aspect of a technical vocabulary is a most delicate matter when separated from the actual use of the terms in question, and especially so when both language and culture have been dead for millenia. The origin of technical terms may have been completely lost from the memories of those applying them, as we have since long forgotten the concrete plumb line behind a term like "perpendicularity". Important parts of the mathematical vocabulary of the Old Babylonian period seems to consist of much older Sumerian terms, some of them taken over as loan-words, some of them translated according to their literal sense. For the former, the etymological origin can be ascribed no argumentatory weigth at all; for the latter, the weigth is dubious when the character of the whole Sumero-Akkadian linguistic merger is taken into account.

Yet, not all indications of an actual concrete (and probably geometric) meaning of the terms can be thus explained away. In some cases, the vocabulary is clearly not a fully stiffened technical arsenal (cf. also pp. 19 and 38). The terms "tear off" and "cut off" are used indiscriminately - and when two Accadian terms with the same suggestive literal sense are used for the same operation, the probability is strongly decreased that this sense has just resulted from the accidents of linguistic development; the "two-sided-ness" of the concepts corresponding to the square root (mitḫartum, ib-si₈) is clearly demonstrated not only by etymology but also by surrounding parts of the text ("the first", "the second"); even the seemingly double use of the terms partly as the root, partly as the square is also only to be conceptually united if side and square are thought of geometrically, as "the square figure" (cf. p. 24). Finally, the exclusive use of the prepositional and adverbial derivations of libbum, "heart", "inside", in identity-conserving additive and subtractive connexions (but never in multiplicative contexts where only ana, ina and itti occur)

At closer inspection, the use turns out not to be completely indiscriminate. In texts where both terms are used (AO 8862, above, pp. 31ff, and IM 52 301, below, pp. 105.43ff) there is a (non-exclusive) tendency to use "tearing" when surfaces are subtracted from surfaces, and to reserve "cutting" to operations on linear extensions (the exception is AO 8862 II, 11, where half the sum of length and width is "cut off" from an "accumulation" of rectangular surface, one half of length and one third of width). This distinction seems to agree with an assumption that the concrete imagery of the two terms was still alive to their users.

indicate that appending and tearing-off are thought of as operations of another sort than multiplication - viz. as operations affecting the "bodies" of those entities to which something is appended or from which something is torn off.

Near the border-line between categorical structure and semantics we could mention the terms "surface", "true"/"false" and "that which is" discussed above: The eqlum, Sumerian a-šà, "surface", is a term for homogenous and inhomogenous second-degree combinations of length and width used in contexts where, in the geometrical interpretation, the combination is in fact to be interpreted as a connected surface; outside the mathematical context it means "field", perhaps mainly considered from the aspect of "something possessing an area"; in mathematical texts dealing with real fields, the areas of these and not the fields themselves are denoted eqlum (VAT 8389 and VAT 8391). Since in more abstract second-degree problems the concrete meaning is in harmony with the delimitation between entities which are called surfaces (viz. those which during the procedure of geometrical solution appear as real surfaces) and entities which are not (cf. p. 35), the categorical and the semantical considerations support each other, and the "surface" must be considered a real geometrical surface. Once we are so far, even the categorical delimitation of "true" and "false" surfaces and lengths fits naturally into the puzzle (cf. p. 35 and note 20).

Concerning the expression "that which is

X", I shall only refer back to the discussion on p. 40f.

When all this positive semantic evidence is taken into account, the information contained in other possibly technical terms ("append", "turn", "raise", "go beyond", etc.), weak and dubious in itself, becomes more weighty because so clearly in harmony with the picture stemming from the other sources.

Another sort of evidence displays itself through the mathematical procedures of the texts. Of such "procedural evidence", three types exist.

In itself, it is no evidence at all that geometrical procedures can be constructed which yield the same intermediate calculations as those found in the Babylonian texts. As long as only positive numbers are involved, any algebraic second-degree-procedure can be reconstructed as geometrical heuristics. What is striking is that the exposition of the procedure in the texts can often be read literally as concrete directives for the geometrical procedure, and that the procedures resulting from the geometrical reconstructions often yield so utterly simple results - see e.g. Figure 11.

Another type is constituted by the rather free "change of variable" practiced by the Babylonians. In a geometrical interpretation this easy change is almost self-explanatory,

since any rectangle or square in a figure can of course be submitted to the usual treatment, irrespective of its status in relation to the original statement of the problem - see e.g. Figure 9, Figure 14 or, for that matter, Figure 11 where suddenly the variable "width" is treated as "the side of my square". If, on the other hand, we exclude the geometrical interpretation, the changes bear witness to a surprising (and, in certain cases perhaps unnecessary) virtuosity of thought.

As were the literal meanings of technical terms, these two types of evidence are weak in themselves and only gain strength because of their harmony with other sorts of evidence. One type of procedural evidence is, however, strong enough to stand on its own feet. Here I think of such remarks in the text which surround the succession of numerical calculations: The distinction between a "first" and a "second" square-root or half occurring in so many texts, and the posing of a preliminary 3 for the width of the rectangle in BM 8390 (see p. 18). They have no meaning outside the geometrical interpretation.

Of the same type although a bit weaker is the evidence of certain errors (or illegitimately abbreviated formulations), e.g. that of YBC 6504, problem 4 (see p. 47) which seem to reflect the pitfalls of a geometric procedure where immediate "seeing" has the status of an argument.

The last category of evidence I labeled "cognitive" for lack of a better term. What I think of is the problem, how mathematics can be thought.

As discussed above, the long successions of numerical calculations in our texts must have been based on some representation giving a meaning at least to a majority of the numbers calculated. We do it most easily by means of symbolic algebra, but that was surely not known to the Babylonians. The Arabic algebras used verbal names, but the predilection for formulations in one variable which is natural to such a medium of thought is no characteristic of Babylonian algebra. Verbal formulations corresponding to the entities occurring during the Babylonian calculations would often be complicated. Furthermore, one would expect them to leave traces in the equally verbal written texts; such traces are, however, absent, except the "false lengths" and "false surfaces" discussed above - expressions which point equally well - or rather better - toward the geometrical procedure.

In general, verbal algebraic arguments are also difficult to make convincing (and the freely varied, normally correct and only rarely erronous procedures show that Babylonian algebra was well argued). Geometry, on the other hand, is much more easily molded as an abstract representation. We know that al-Khwârizmî in the beginning of his Algebra tried to demonstrate various algebraic transformations

(over and beyond the solutions of the second-degree equations) by means of geometric heuristics (see Rosen 1831: 31-34). Bhaskara II claims that geometrical demonstrations of algebraic rules go back to "ancient teachers" (Datta & SIng 1938:4), and, in fact, his much quoted heuristic figure illustrating the Pythagorean theorem followed by the only explanation "see!" (e.g. Cantor 1907:656; Heath 1956:I, 354; Juschkewitsch 1964:99) may in fact, with many geometrical justifications of a similar sort, go back to the súlba-sûtras. In these, many procedures similar to those ascribed above to the Babylonians can be found (see e.g. Âpastamba-Śulba-Sûtra II-III, Bürk 1901: II, 331-337). Also in Han-China, manipulations of geometrical figures seem to have been the method behind the solution of second-degree equations (see Vogel 1968:135f), and in the slightly younger earliest Chinese proof of the Pythagorean theorem, our Figure 13C is found as the foundation of an accounting argument ($c^2 = 4\frac{a\cdot b}{2} + (a-b)^2 = a^2 + b^2$ - Juschkewitsch 1964:62).

So, a priori, even before any internal evidence of the texts is investigated, it would appear cognitively and historically probable that Babylonian algebra made use of geometrical heuristics - it would constitute a queer historical exception among early algebras if it did not. Only our own complete addiction to arithmetical symbolic algebra has made us blind to that.

Addendum: *Anbouba (1978:76) points to a place where al-Khwârizmî does not refer explicitly to the use of geometrical reasoning (p. 63f in Rosen's translation), but where the style and a parallel passage by Abû Kâmil shows the argument to be built on geometrical heuristics. By al-Khwârizmî as well as by Abû Kâmil (whose solution is translated in Levey 1966:105ff), first a general rule is set out, and next a specific example is presented. This second part transforms the problem into one of rhetorical algebra ("square plus things ...), but apart that it is closely analogous in style to the Old Babylonian problem texts. The first part by Abû Kâmil, on the other hand, is precisely the sort of didactical demonstration which one must imagine by the Babylonians if the assumption of a geometrical heuristic is accepted.*
The "cognitive" attraction of such geometrical arguments is demonstrated by the fact that Abû Kâmil's argument is taken over directly by Leonardo Fibonacci (see Boncompagni 1857:413) and from him by John of Murs in the Quadripartitum numerorum (L'Huillier 1980: 204).

After this summary of the terminological, the procedural and the "cognitive" evidence in favour of the use of geometrical heuristics in Babylonian second-degree algebra, I will permit myself to consider the case decided for the time being.

THE FIRST DEGREE

In order to put the geometrical heuristics into a total perspective, I shall now consider some select problems and algebraic transformations of the first degree. To which extent can they be said to be similar to those methods and problems which we have discussed already?

Let us first consider a geometric problem, viz. Strasbg. 367 (MKT I, 259f; cf. Fig. 20): A trapezoid is cut into two parts, ABEF and BCDE in Figure 20A, the surfaces of which are respectively 13`3' and 22`57'. Besides, the "lower length" AB is one third of the "upper length" BC, and the sum of the respective differences between widths (HF and GE, bold-lined in the figure) is given to be 36.

First, by "posing" 1 and 3, [HF and GE] are found as $1\cdot\{(1+3)^{-1}\cdot 36\}$ and $3\cdot\{(1+3)^{-1}\cdot 36\}$, obviously by some reasoning involving a kind of proportionality. The multiplications involved are all designated by the term n i m, a Sumerian term for "high", "lift up" etc. - i.e. by an semantic equivalent of

***Addendum**: Cf. full translation and analysis in the appendix. At closer inspection, the text contains some clues which I did not notice at the first reading. (See pp. 105.34ff).*

našûm~íl, "to raise"[25].

The following step is a division of the "lower surface" by 3 ("20′ ana ...n i m"). This suggests a scaling in one dimension, corresponding to the interpretation shown in Figure 20B, where the lower surface is replaced by BC′D′E - the operation of Figures 4B and 18B. (A somewhat related interpretation is given in Vogel 1959:74). In the next step, the difference between the surfaces ABEF and BC′D′E is found, a difference which, in two different representations, is shown in Figure 20C. That something rather concrete, like one of these figures, is thought of, follows from the next step, where 1 and 3 are added, the half broken from the sum, and the resulting 2 divided (by multiplication by 30′) into the difference, the outcome Δ = 2`42' being designated "false NU". These steps would, as the distinction between "breaking into halves" and multiplication by 30′, be unnecessary complications if only abstract relations of proportionality and no concrete representations had been thought of. Δ is the surface of the rectangle HEKF, and so the "false NU" may remind of the "false" auxiliary rectangle of VAT 7532 (see p. 52, and Figure 17) - and indeed, NU may be an ideogram for ṣalmum, "picture", "representation", "monument", "figure", "form".

In fact, the operations are (inversed) those of the algorithm used to calculate the area of a trapezoid with parallel widths 1 and 3. So, Δ must probably be the length of a trapezoid; in any case, the interpretation as the area of a rectangle seems impossible. Cf. the appendix, pp. 105.39f.

From Δ, the length r of our auxiliary rectangle is found by a rather curious procedure: From some mathematical equivalent of Δ = 9·r (not stated explicitly), first [r^{-1}] is found as 9/Δ (by means of the "posing"-construction, although 2`42` (=2·3⁴) is a regular number), and then r is found as $(9/\Delta)^{-1}$. No obvious reason for such a route is at hand, but perhaps this passage of the text may be taken as an indication that the Babylonians did not think their arguments quite the way we do.

From r, the upper and lower lengths are found by multiplication by 1 and 3 - and so, r, which as so often has no name, is not considered identical with the upper length AB (cf. BM 13901, no. 10 and 14, p. 20ff, where the same "false position" was made). Finally, it appears, the area of the triangle DFJ is calculated and subtracted from the total area[26], leaving the rectangle ACDJ, of which the length AC is now known, and hence also the width (calculated this time directly, not via its reciprocal). Finally the remaining dimensions of the original trapezium are found: AF, BE and CD.

Addendum: For the subtraction, the term du₈ ≈ paṭarum is used, while in other places of the same tablet is occurs (as usually) when a reciprocal is found. The same double use is found in Strasbg. 362 (closely related, but in a slightly deviating language which suggests another scribe and perhaps a slightly later origin). This suggests that the term is (at least in these presumably rather early texts) to be understood as "undo" or "detach", and not *as "find out" (the term used in my translations) or "solve" (as a problem is solved): Finding the reciprocal of n consists in detaching the n'th part from 1 (or 1`).*
The non-validity of the interpretation "find out" (which was, e.g., put forward by Thureau-Dangin - 1936:56) is demonstrated by IM 52 301: There, reciprocals are found by paṭārum, while square-roots ("side") are to be asked for (šâlum). The two processes are clearly kept apart. Cf. also YBC 4675, obv. 15 (below, p. 105.25) and Db₂-146,9,12 and 22, where the "side" is "taken" (laqûm) (the latter text was published by Baqir in Sumer 18 *(1962), 12f).*

All in all, we seem to be confronted with a procedure which follows the usual geometrical argumentatory pattern. Not only is a geometrical interpretation possible, but the resulting procedure is intuitively simple, and, more decisively, the hitherto unexplained "false NU" becomes explainable, as do the calculatory steps surrounding it.

Still very concrete, but seemingly more verbal in its way of reasoning is the treatment of a set of problems concerning two fields (VAT 8389 and VAT 8391, MKT I, 317-323.

All cases deal with two fields I and II. The yielded rent by the first one is 4 gur (1 gur = 5` qa) of grain per bur (1 bur = 30` sar), while that of the second field is 3 gur per bur. In one of the problems, which may probably be regarded as a didactical prolegomenon to the whole group, the two surfaces S_I and S_{II} are also given, while in four others, the four combinations of $S_I \pm S_{II}$ and $Y_I \pm Y_{II}$ (Y = total yields) are given.

In all cases, the specific yields of the two fields are recalculated in terms of the basic units of the mathematical texts, as $20`\cdot(30`)^{-1} = 40'$ [qa/sar] and $15`\cdot(30`)^{-1} = 30'$[qa/sar]. These specific yields are spoken of as "false grain". In the prolegomenon, the yield of each field separately and finally the total yield is then calculated. In those of the others where $S_I + S_{II}$ is given, it seems that first a hypothetical situation with two equal fields I′ and II′ is considered (cf. Figure 21A). The value of $Y'_I \pm Y'_{II}$ corresponding to this situation is calculated, and subtracted from the real value. The increase of the sum/difference of the total yields, each time an extra sar is transferred from II′ to I′, is found, and by division the total transfer needed in order to obtain the total sum/difference is found[27].

Addendum: So, we may perhaps imagine that the specific yields were thought of as "the yield of the first/second field, had its surface been 1 sar".

As always, it is impossible to demonstrate from the numbers of the calculation that this was the argument. But the harmony with the concreteness of thought of the geometrical heuristics supports the interpretation, as does in a curious way the two remaining problems.

In these, $S_I - S_{II}$ is given. In both cases, it appears that the contribution of I′, that part of I which "goes beyond over" II, to $Y_I \pm Y_{II}$ is calculated (cf. Figure 21B). In the case "$Y_I - Y_{II}$ given", the contribution to the residual difference, each time a sar is given to II, and a corresponding sar to I″, is found, and by division the surface of II′ and, correspondingly, the surface of I″ - whence also the surface of I.

In the formulation of this, a weak support for the concrete argumentation is found. Indeed, when the total extra surface is found, this is at once stated to be the surface of the first field (while it is the surface of II, the "second" field of the enunciation). If we remember the consequence with which in Strasbg. 367 (see p. 75) r was multiplied <u>first</u> with 1 and <u>next</u> with 3. and how later the width of the rectangle ACDJ was added first to the upper width, then to the transversal, and only at last stated to <u>give</u> (sum,-i.e. not to <u>be</u>) the lower width (a pattern which corresponds to many other texts), the change of order for the two fields indicates that precisely S_{II}, and no other entity of the same magnitude, had really been thought of during the argumentatory process.

More decisive is, however, the support

lended by the case "$Y_I + Y_{II}$ given". Here, the residual of $Y_I + Y_{II}$ is accounted for in a remarkable way (cf. note 7): 1 the <u>wāṣûm</u>, a unit area, is split into two halves, and its yield is calculated as if one half belonged to I, the other to II. That this happens is not a conjecture dependent on interpretation, but a pure consequence of the calculations; only the concrete explanation, that the <u>wāṣûm</u> can be thought of as a "normal" sar of the remaining surface, equally composed from the two types of soil, is an interpretation,- but an interpretation which it is difficult to do without (Neugebauer is perceptibly uneasy without it - MKT I,333). Finally the total remaining area is found by division - and by an argumentatory short-circuit due to an accidental coincidence of numbers, it is claimed to be S_I.

As explained in the revised note 7, the "unit area" is probably only a <u>functional</u> interpretation of the term wāṣûm, while the semantics may be the <u>width</u> of the unit area. For the present purpose, this is unimportant.

It will be observed that the interpretation in question is a natural extension of the understandind of the "specific yield" suggested by the term "false grain" - cf. addendum p. 76.

A final problem of the same tablet dealing with the same fields I have left out. Here, $Y_I + Y_{II}$ and S_I are given. It is so simple that a distinction between more or less concrete patterns of argument are out of place.

For the others, however, the total picture emerging is one of very concrete reasoning. Since everything deals with real surfaces of fields, these of course play an important role as providers of names for the quantities calculated ("the false grain" of this, and "the total grain of that"). But the actual heuristic argument seems to be verbal and conceptual. No traces of real geometrical arguments can be traced in the text and, very indicative, these tablets belong to those which ask the performer to keep intermediate results "in your head".

Before we leave these texts, a point of some terminological importance will be noticed. VAT 8389,obv.II.6-9 deals with the division by an irregular:

6. How much to 1°10″ shall I pose
7. which 5`50° that your head retains, gives me?
8. 5` pose. 5` to 1°10′ raise,
9. 5`50° it gives you.

The same construction is found in VAT 8520, Obv. 24f (translated next page).

This shows that the term "to pose" always used in this context is not in itself considered a multiplication, since it is followed by one (the "raising"). Accordingly, in other cases where no explicit mention of a multiplication follows the posing, the multiplication must be regarded as automatically implied by the term, the context or the act; posing itself can be no technical term for a concept or a procedure of multiplication.

That the juxtaposition of "posing" and "raising" is no scribal error or accidental pleonasm appears from the strictly parallel passage VAT 8391, rev.I.28-30.

Purely arithmetical manipulations in the context of one of the abstract lengt-width-surface-problems seems to be present in VAT 8520, problem 1 and 2. Problem 1 (MKT I,346f) runs as follows[28]:

'Accumulated" translates Obv.1. *akmartum < kamārum.*

1. The 13th part of the accumulated of igûm and igibûm
2. until 6 I have "doubled", from inside the igûm
3. I have torn off: 30′ I have left. 1 the surface. Igûm and igibûm what?

'This restitution follows ⇐ MS, p. 115).

4. Since »the thirteenth part of the accumulated of igûm and igibûm
5. which until 6 i have doubled and from inside the igum
6. I have torn off: 30′ I have left« he said,
7. 13 of the thirteenth part pose; 6 (until) which he "doubled" pose;
8. 1 the surface pose; and 30′ which he left pose:
9. From 13 of the thirteenth part, 6 (until) which he "doubled"
10. tear off, 7 you leave
11. 7 which you left and 6 [until] which you "doubled"
12. that your head retain!
13. 7 to 6 raise, 42 to 1 the surface raise, 42.
14. 42, that your head retain!
15. 13 of the thirteenth part to 30′ which he left
16. raise, 6°30′ to two break: 3°15′.
17. 3°15′ together with 3°15′ give reciprocally: 10°33′45″.
18. To 10°33′45″, 42 which your head retains
19. append: 52°33′45″.
20. The side of 52°33′45″ what? 7°15.
21. 7°15′ and 7°15′ its equal lay down/draw:
22. 3°15′, the thing which was "given", from one tear off, to the other append.
23. The first 10°30′, the second 4.
24. What to 7 which your head retains shall I pose
25. which 10°30′ gives me? 1°30′ pose, 1°30′ to 7 raise, 10°30′
26. it gives you. 1°30′ which was posed is the igûm.

27. The reciprocal of 6 which your head retains, find out: 10′.
28. 10′ to 4 raise, 40′ the igibûm.
29. Since 1°30′ the igûm, 40′ the igibûm, what the surface?
30. 1°30′ the igûm to 40′ the igibûm raise, 1 the surface.
31. 1°30′ the igûm and 40′ the igibûm take together: 2°10′.

Rev. 1. The thirteenth part of 2°10′ what? 10′.
2. 10′ until 6 double: 1 from 1°30′
3. of the igûm tear off: 30′ you have left.

The text deals with the igûm and the igibûm, a pair of complements from the table of reciprocals. Symbolically, we can represent the problem thus (x=igûm, y=igibûm):

$$x - 6\cdot\frac{1}{13}(x+y) = 30' \qquad x\cdot y = 1 .$$

The first equation is transformed into

$$(13-6)x - 6y = 13\cdot 30', \text{ i.e. } 7x - 6y = 6°30'$$

The second equation gives

$$7x \cdot 6y = 7\cdot 6\cdot xy = 42\cdot 1 = 42 .$$

So, we now have a standard-problem

$$X - Y = A,\ X\cdot Y = B \quad (7x = X,\ \ 6y = Y),$$

and in the normal geometrical way we find

$$7x = 10°30' ,\quad 6y = 4$$

whence $x = 1°30'$, $y = 40'$. Finally follows a proof.

The solution of the standard-problem is closely parallel to the formulations of the same procedure discussed above (e.g. YBC 6967, obv. 5 onwards, cf. p. 11 and Figure 3) and teaches us nothing new. Remarkable is, however, the initial part of the solution, where the linear "equation" is transformed. There is no trace of a geometrical imagery in these

lines. On the contrary, the identifications of the numbers ("13 of the thirteenth part"; "6 which doubled", etc.) refer to the arithmetical roles of the numbers. Neither is there in the corresponding transformation of the second-degree equation anything which suggests a geometrical representation - 7·6 is found by "raising", not by "giving reciprocally" (while this term is found in Obv.17). Finally, numbers are to be kept in head, and are neither "posed", "inscribed" nor "laid down"/"drawn". All in all, everything points to the conclusion that the whole linear transformation was made by verbal, arithmetical reasoning.

Even the way in which x is found from 7x (and y from 6y) suggests arithmetical reasoning. 7 is still something which is kept in the head, and not referred to by any possible palpable representation or role. Only the central portion of the text is, if not necessarily drawn, at least thought of in the imagery of a drawing (cf. p. 28) - the latter possibility is favoured by the occurrence of the "head" in line 18 as the carrier of 42 [the surface of the gnomon].

Two terminological points are of interest. Firstly, that in obv.25, the division by the irregular number 7, once more the "posing" and the "raising" are juxtaposed - cf. the discussion of the implications of this on p. 79. Secondly one observes the occurrence in this text of both "reciprocal giving" and "raising" when surfaces of rectangles are calculated (obv.17, obv.30). The first instance is one where geometrical imagery is to be expected, while the second is the proof where pure calculation without any visual imagination could do. This suggests a conceptual distinction: according to this, "reciprocal giving" is thought of as the <u>construction</u> of the rectangle, which secondarily of course entails the emergence of a surface equal to the product; "raising", the term also used in connexions involving considerations of proportionality (geometrical and non-geometical alike), designates or emphasizes the calculation of the area as a product.

This could be put more sharply: "reciprocal giving" is spoken of when a construction is to be performed, when the surface is "built". "Raising" occurs when the rectangular surface is already there, and only the area is to be calculated. Precisely the same distinction is seen to be respected in VAT 8390 (see above, pp. 16f, where the first problem of the tablet is translated; the second problem is parallel even in the finest terminological details).

Cf. the exclusive use of "raising" for the calculation of the areas of triangles, trapezoids and irregular quadrangles, and the marginal note above, p. 38, on double constructions involving "reciprocal giving" and a-rá.

We have already met with another text where both terms coexisted, <u>viz.</u> VAT 8390 problem 1 (see p. 16-19). The distinction here suggested agrees perfectly with the occurrences of the two terms in this text, to an extent which even supports its geometrical interpretation. So, the distinction occurs to be a systematic one, and the agreement of the two texts provides further confirmation that "reciprocal giving" should always be read geometrically, as construction, "building"[29].

In the siege ramp-calculations in BM 85194 and BM 85210, "raising" was also used in cases where areas of rectangles were calculated - cf. p. 62.

Seemingly non-geometrical reasoning in a definitely non-geometric context is found in the initial part of that commercial problem by which we started our investigation (Texte XIII, TMS p. 82; improved readings in Gundlach & von Soden 1963:260f). The text can be so translated:

A strictly analogous problem is YBC 4698, obv. II.12-19 (cf. Friberg 1982:57).

1. 2 <u>gur</u> 2 <u>pi</u> 50 <u>qa</u> (=12`50˙ qa) of oil have I bought. From what was bought for 1 šekel of silver
2. 4 <u>qa</u> of oil each (time) I have cut away.
3. 2/3 <u>mina</u> (=40 šekel) {...} of silver as profit have I seen. How much (for each šekel)
4. have I bought and how much have I sold?
5. You, 4 <u>qa</u> of oil pose and 40′ <u>mina</u> the profit pose.
6. The reciprocal of 40 find out, 1′30″ you see, 1′30″ to 4 raise, 6′ you see.
7. 6′ to 12`50˙ of oil raise, 1`17˙ you see.

8-12, (see p. 7).

12. ... 11 <u>qa</u> for each (šekel) have you bought 7 <u>qa</u> have you sold.
13. Silver how much? How much to [11 <u>qa</u> shall I p]ose
14. which 12`50˙ the oil gives me? 1`[10˙ pose, 1 <u>m]ina</u> 10 šekel si[lver].
15. At 7 <u>qa</u> each which you s[ell, the oil]
16. of 40 (šekel of) silver how much? 40 to 7 [raise],
17. 4`40˙ you see. 4`40˙ (<u>qa</u>) oil [the profit(?)].

In order to be in harmony with the questions of the text, I fix the orders of magnitude of the numbers in agreement with a choice of <u>qa</u> and šekel as units. Gundlach & von Soden use <u>pi</u> and <u>mina</u> instead. The final results are of course the same.

A damage in the lower right corner of the tablet has given rise to a number of lacunae (which, for the relative ease of reading I have emendated in agreement with Gundlach & von Soden). Irrespective of emendations, what remains in these last lines provides an important clue to the procedure of the beginning.

The problem is the following: 12`50˙ <u>qa</u> of oil are bought at the rate of x <u>qa</u> per šekel and sold at the rate of y <u>qa</u> per šekel; the profit of the whole transaction is 40 šekel.

The difference between the rates is x-y = 4 qa. As it is seen from the lines 8-11, the problem is solved via the recognition that the product of the rates amounts to the difference between the rates divided by the total profit and multiplied by the quantity of oil:

$$x \cdot y = 12`50' \cdot \frac{4}{40} .$$

How could this relation be found? It seems far from self-explanatory.

The "proof" of lines 13-17 provides the clue. Indeed, it is no proof at all, as it does not control the profit. Instead, it calculates, first, from the rate of purchase, the amount of silver invested in the transaction; second, from the rate of sale, the amount of oil which gave rise to the profit. Since neither quantity occurs in or is asked for in the enunciation of the problem, their interest can only be explained by their presence in the argumentation leading to the solution.

If the solution has gone via these two quantities, it will have run approximately like this:

If the oil was bought at some rate, the number of šekel invested has to be the ratio between 12`50' qa and this rate ($\frac{12`50'}{x}$). For each šekel, I gain a profit of 4 qa of oil, and thus a total profit of 4 times the ratio between 12`50' and the rate of purchase. This is equal to the amount of oil sold at the rate of sale for a total amount of 40 šekel, i.e. the rate of sale times 40 šekel ($40 \cdot y$). Anybody able to command the relations between these ratios and products (and nobody else) will from this be led to the above relation.

The Babylonians (or rather, their Susian followers) did find the relation, and so their mastery of such first-degree problems involving several unknowns in composed products must have been sufficient. Since everything is formulated briefly and arithmetically, with no hint of a non-verbal representation

(whereas geometrical suggestions abound in the central section dealing with the second-degree problem), we may suppose once again a verbal, non-geometrical reasoning for the first-degree sections.

A MATHEMATICS LESSON

Analyses of procedures, formulations, vocabulary, errors etc. are useful tools if we want to penetrate Babylonian mathematical thought. Still, the evidence provided by these sources is bound to be of an indirect nature, and thus only convincing because of its astonishing coherence. So, a direct report of a few Babylonian mathematics lessons would constitute an invaluable support for the investigation.

If no complete report exists, at least two texts have been published which contain not just the enunciation of a problem or the enunciation followed by a description of the procedure, but also the didactical explanations which the teacher would give/had given of the transformations of his "equations".

The first of the texts in question is Texte XVI of the Susa texts (TMS p. 91f). Since my interpretation of the character of the text differs rather much from that given by the editors (TMS p. 93ff), we shall need the text:

A.1. The 4th of the width from length and width I have torn off, 45. You, 45

2. to 4 raise, 3` you see. 3`, what (is) that? 4 and 1 pose,

TMS transcribes the beginning of line 3 as "[50 ù] 5 ZI.A(!) <GAR>" and interpretes ZI as a (phonetically motivated) writing error for SI, which would give the passage the meaning "50 and 5 which go beyond <pose>". However, the A is damaged and clearly separated from the ZI. As far as I can see on the autography, the traces might as well represent the lacking GAR, which would give the reading "[50 ù] 5 zi gar", "50 and 5, the torn-off, pose". This has the clear advantage over the reading of TMS to be in agreement with the zi, "to be torn off", of line 4, as well as with those of line 1, 5 and 8.

On obscure terms in line 7:
α. manātum: AHW lists only this and another, very obscure occurrence, equally from TMS, and suggests hypothetically an identification with Hebrew and Aramaic m^enāt, "Anteil". HAHw (438²-439¹) exemplifies this term by "Anteil der Priester u. Leviten" and "d. Teil (Beitrag) des Königs". The ensuing "share/contribution of the width" fits the present text excellently.
β. The reading "retain" is a conjecture (ki:il!) due to von Soden (1964:49). TMS has ḫulum, Assyrian for "way", interpreted as "method" by the editors.

Observe that 4x-(4y-1y) is calculated instead of 4x-4y+1y.

3. 50 and 5, the torn-off, pose. 5 to 4 raise, 1 width. 20 to 4 raise,
4. 1`20` you see, 4 widths. 30 to 4 raise, 2` you see, 4 lengths. 1 width to be torn off,
5. from 1`20`, 4 widths, tear off, 1` you see. 2`, lengths, and 1`, 3 widths, take together 3` you see..
6. The reciprocal of 4 find out, 15′ you see. 15′ to 2`, lengths, raise, 30 you see, 30 the length.
7. 15′ to 1` raise, 15 the count (manātum, abstract noun from manûm, "to count") of the width. 30 and 15, retain(?).
8. Since the 4th of the width, to tear off, he said to you, from 4 1 tear off, 3 you see.
9. The reciprocal of 4 find out, 15′ you see, 15′ to 3 raise, 45′ you see, 45′, the same as (there is) of widths, pose.
10. 1, the same as of lengths, pose. 20 the true width take (leqe, from laqûm, "to take"). 20 to 1 raise, 20 you see.
11. 20 to 45′ raise, 15 you see. 15 from 30 15 (an additive writing of 45) tear off,
12. 30 you see, 30 the length.

B.13. The 4th of the width to that by which length over width goes beyond, I have appended,
14. 15. You, 15 to 4 raise, 1` you see, what (is) that?
15. 4 and 1 pose. {15 to 4 raise, 1` you see, what (is) that?}
16. 15 scatter[30]. 10, that which goes beyond, and 5, that which was appended, pose. 20 the width
17. to 10 that goes beyond append, 30 the length. 20 the torn-off pose. 5 to 4 raise,
18. 20 you see. 20 the width to 4 raise, 1`20` you see.
19. 30 the length to 4 raise, 2` you see. 20 the width
20. from 1`20` tear off, 1`[...] 1`you see [...]
21. from 2`, the lengths, tear off, 1` you see. What (is) that?
22. From 4 of the fourth 1 tear off, 3 you see. ~~The reciprocal of 4 find out, 15′ you see.~~
23. 15′ to 3 raise, 45′ you see, the same as [there is] of widths, pose. 4 tear off (?!)[31].
24. 1, the same as of lengths, pose. 1 take to 1 length,

25. 45′ to 1 width. 20 the width, 20 to 45′ raise,
26. 15 you see [...] 15 to 15 append, 30 you see, 30 the length.

The text deals with two relations, viz. (if as usually we put length=x, width=y)

$$(x+y) - \frac{1}{4}y = 45$$

and

$$\frac{1}{4}y + (x-y) = 15.$$

Obviously, none of these relations suffice to determine x and y. But in both sections, y is presupposed to be 20. This leads Bruins to regard the two sections as consisting each, partly of a discussion, partly of a solution (TMS, pp. 93-95). However, x is also referred to as a known number, ex- and implicitly, both in the initial discussion-parts and in the supposed solutions. So, the texts give little meaning when so read.

Instead, we may look at them from the sole point of view of the discussion. Section A (the obverse, lines 1-12), begins by a multiplication of 45 by 4 (corresponding to the 4th), and asks for the meaning of the result - i.e., what is looked for is a conceptual representation of the number 45·4 = 3`. In line 3 we see that both x+y = 50 and the 5=¼y (to be torn off from 50) are known. So, we must imagine the teacher explaining that a length of 30 and a width of 20 are given. These values imply, he shows, that $(x+y) - \frac{1}{4}y = 50 - 5 = 45$. In order to explain the significance of the 3`, he multiplies all the components of 45: 4·5, 4·20,

Cf. IM 52 301, problem I - see below, p. 105.43, marginal note.

ADDENDUM

At the workshop, Peter Damerow suggested a schematic clarification along lines similar to these:

α		1 x	+	1 y	-	¼ y	=	45
β		1 x	+	45′ y			=	45
γ	1	30	+	20	-	5	=	45
δ		50			-	5	=	45
δ'		30	+	15			=	45
ε		4 x	+	4 y	-	1 y	=	3`
ζ	4	4 x	+	3 y			=	3`
η		2`	+	1°20′	-	1	=	3`
ϑ		2`	+	1`			=	3`

Apparently, the "1" and "4" posed in line 2 of the text are the factors written to the left of the two groups of equations. The rest discusses the relations between the lines α to ϑ.

It is seen that α represents the original equation of "lengths" and "widths", written symbolically, while ε is obtained from this original equation by a multiplication by 4. γ and η represents the same equations when the known values of length and width are inserted.

In the text, line 3 "poses" the 50 and 5 of γ, representing 5 as "that which is torn off" (from 50). Next (line 3-5), the transformation of γ into η is explained term for term in order to solve the problem raised in line 2, which meaning to ascribe to the 3` which arise when the right-hand side of α is multiplied by 4. This is done with reference to ε, ζ and ϑ.

Line 6-7 explains the reverse transformation η to γ, referring to δ', where the respective contributions of lengths and widths are separated. Line 8-12, finally, explains δ' in terms of β where the coefficients of x and y, i.e. "as much as there is" of lengths and widths, is found and multiplied to the numerical value of these entities.

So, there is a far-reaching analogy between the Babylonian text and our own treatment of such a problem. The coefficients are spoken of explicitly, and the factor by which the equation is multiplied in order to simplify is mentioned almost explicitly.

4·30, and the meaning of each term is explained (1 width, 4 widths, 4 lengths), and finally the meaning of 3` as the accumulation of 4 lengths and 3 widths is found.

Next, everything is multiplied by 4^{-1}; 2` which represented 4 lengths becomes 30, the length. 1` which represented 3 widths becomes 15, the "count" of the width, i.e. the numerical contribution of widths to the initial 45.

After this deep-going discussion of the significance of everything, the teacher exposes once again (lines 8-12), more briefly but according to the principles just made clear, how the equation can be understood as a sum of contributions from length and width (i.e. given a form corresponding to our reduction ax+by). $1-\frac{1}{4}$ is calculated as $(4-1)\cdot 4^{-1}$, and so the numbers of lengths and widths are found to be 1 and 45′, respectively; when 45` widths (=15) are subtracted from the 45 (written already as a sum of the two contributions), the remainder is seen to be 30, the length, as it should be.

Cf. ***addendum*** *p. 37.*

Section B of the tablet (lines 13-26, the reverse) follows the same scheme, only this time the combination given of length and width is the sum of "that which goes beyond" (x-y=10) and the fourth of y (=5, referred to in line 16 as "that which was appended"). Once again, the initial deepgoing and very concrete discussion is followed by a second more brief and abstract treatment of the equation where the understanding supposedly acquired by the initial dissecation of everything is now at hand as a cognitive background.

A number of points appear from the text:

- Most important is perhaps the corroboratior of our general assumption, that the numbers of the mathematical texts must have possessed a conceptual significance, must have corresponded to a mental or a physical representation.

- In the final transformation of the equation, an analogue of the concept of a coefficient appears, designated "the same as there is of".

Cf. p. 27, on BM 15 285, No 10, the expression "16 of a square figure", 16 mitḫartim. This analogue, as well as grammatical considerations, suggests the genetive form "of length(s)" etc. - the writing itself is ideographic without grammatical complements.

- This coefficient is no abstract multiplier, but is spoken of concretely. It is the outcome of an accounting procedure.

- At a number of points it occurs that the objects counted are not just the length and the width, but rather 1 length and 1 width: lines 3 and 4, and especially 10-11 and 24 where, in the first case, from <u>the width</u> of 20 <u>one width</u> of twenty and hence 45′ of <u>this one width</u> is calculated, while in the second case 1 is explicitated as the number of "one length"s. If this shall have a meaning, it must imply that other entities could also occur as objects of the accounting - cf. BM 13 901 no 11 (p. 20), where one seventh of the side was (with

its "square figure") the unit of accounting, and YBC 6504, no. 3 (pp. 42, 46), where x-y played a corresponding role[32].

A didactical exposition like that of this text can be made verbally, or by writing-down numbers. It can also involve reference to drawings of lines representing the quantities in question. The possibility that other units of accounting than length and width themselves could be used suggests that the latter possibility may conform to truth. Further evidence in the same direction is made up by some details in the vocabulary - especially the identification (in line 17) of 20 as "that which was torn off" (when x-y was formed) and not as "the width" (an identification which is regarded as absurd by the editors of the text - TMS p. 93), but also the occurrence of a "true width" in line 10 (regarded as a scribal error in TMS, and indeed rather meaningless unless the assumption of a geometrical representation is taken into account).

***Addendum:** To avoid misunderstandings: Such drawings of lines would of course not serve as proof or justification, not even in the heuristic sense - that task is taken care of by the verbal exposition. They would just serve as manifest representations of the quantities spoken of - as the lines used in the theory of proportions in the <u>Elements</u>.*

The "posing" of the coefficients of length and width (in lines 9-10 and 23-24 may also be an indication that some non-mental medium was used where numbers could be posed.

A text of similar concerns is Susa-text no. IX (TMS pp. 63f). Unfortunately, it it rather damaged, so only the acquaintance with other related text allows us to follow it.

The text contains three section A, B and C, of which the first two are expositions of a technique of transformation, while the third uses the technique to solve a problem:

In both A, B and C, "taking together" is a translation of UL.GAR, cf. TMB p. 240. Especially VAT 6598 Rev. I.5 (TMB: no. 231 l. 5) demonstrates ideographic equivalence between UL.GAR and kamârum.

A.1. The surface and 1 length I have taken together, 40′...

2. As 1 length to 10′, the surface ...

3. Ú.UL (functionally analogous to "pose") 1 as extension (KI.GUB.GUB, the translation derives from line 32) to 20′, the width

4. Ú.UL 1°20′ to the width which 40′ together with (i.e., which together with the length gives 40′)

5. Ú.UL 1°20′, together with 30′ the length, turn (into a frame - cf. n. 3a), 40′ its name (šumum - cf. p. 86).

It will be observed that the result of a "turning"-procedure is something possessing a number as its "name" - not just the number itself. This agrees well with the geometrical meaning given to the group of terms "turning", "reciprocal giving", íb-si₈, etc. It is also in harmony with the habit to use a number as an identification ("30 the length", "15 the thing given", etc.).

6. When, in this way, to 20′ the width, as he said to you,

7. 1 you have appended, 1°20′[33] you see. From here

8. you search (from šâlum, ask, search, research, investigate). 40′ the surface, 1°20′ the width, the length what?

9. Such the being-made (nēpešum)

B.10. [... (surface, length and width)[34] taken] together: 1. By the Akkadian

To be quite correct, the term nēpešum should be rendered the "having-been-made", both in order to render the perfective aspect inherent in the form and in agreement with the place where the term is used: Invariably after a procedure has been described.

11. ... 1 to the width append. When 1 to the length you have appended,

12. ... append. 1 and 1 turn into a frame, 1 you see,

13. [... (to the 1, accumulation of length)] width and surface append: 2 you see.

14. ... that to 30′ the length 1 you append, 1°30′.

15. [... (surface???)] of 1°20′, width, of 1°30′ length,

16. ... give reciprocally, what its name?

17. 2 the surface.

18. Such the Akkadian.

C.19. Surface, length and width I have taken together, 1 the surface. 3 lengths, 4 widths I have taken together,

20. [... Its 17]th part to the width I have appended: 30′.

21. You, 30′ until 17 times go: 8°30′ you see.

22. [(to 17 widths)] 4 widths append, 21 you see.

23. 21 as much as of widths pose. 3 of the triple length,

24. 3 as much as of lengths pose. 8°30′ what its name?

25. 3 lengths and 21 widths taken together

26. ... 8°30′ you see,

27. 3 lengths and 21 widths taken together.

28. 1 to the length append and 1 to the width append, turn into a frame:

29 1 to the accumulation of surface, length and width append, 2 you see.

30. ... surface. When the length and width of 2 the surface,

31. [...(1°30′ the length) together] with 1.20′ the width give reciprocally,

Addendum: wa-şu-bi could, as far as I can see, either be the genitive of a noun waşūbum derived from waşābum, the "extension" proposed. Or it could belong to an adjective derivation of the same verb, waşubum. In this case, the sentence should mean "1 of the extended length and 1 of the extended width" - and that is mathematically and textually meaningless, since the extended lengths and widths are, respectively, 1°30′ and 1°20′, and since they are spoken of in line 31 without the epithet waşubum.

32. [... 1 of the exten]sion (wa-şu-bi, from waşābum, " to append") of the length and 1 of the extension of the width

33. ... take together, 2 you see.

34. [... (3 extended lengths and 21 extended widths)], 32°30′ you see.

35. In this way you search.

36. ... the width until 21 take together,

37. ... to three of the lengths raise

38. ... to 2, of the surface, raise.

39. [(2`6′you see) ... (To two)] 32°30′the accumulation break, 16°15′ you see.

40. [... (16°15′and)] 16°15′ its equal (DUH[35]) pose, turn into a frame,

41. 4`24°3′45″ you see. 2`6′ ...

42. from 4`24°3′45″ tear off, 2`18°3′45″ you see.

43. What the side? 11°45′ the side. 11°45′ to 16°15′ append,

44. 28 you see, from the second tear off, 4°30′ you see

45. The reciprocal of 3, of the lengths, find out, 20′ you see, 20′ to 4°30′,

46. {20′to 4°30′} raise: 1°30′ you see.

47. 1°30′ the length of 2 the surface. [... (What shall I pose)] to 21 of the width,

48. which 28 gives [me? ...] 1°20′ the width

49. of 2 the surface. Again, 1 from 1°30′ tear off,

50. 30′ you see. 1 from 1°20′ tear off,

51. 20′you see.

A deals with the relation

$$x \cdot y + x = 40'$$

in a rectangle where $x = 30'$, $y = 20'$. The text discusses the transformation to the form

$$x \cdot (y+1) = 40',$$

via the extension of x by 1. As in Texte XVI which was just discussed, the values of x, y and xy are all supposed to be known, and used (exception made of x which is unnecessary) to explain the meaning of the procedure - cf. the way $xy=10'$ occurs in line 2. What comes out is a meaning which corresponds very well to Figure 22A. One will notice the familiar character of the procedure, which was also used in AO 8862, problem 1 and 2 (see pp. 30-39, and Figures 9 and 10). So, the interpretation of the procedures used there (addition of 2 to the width, subtraction of $\frac{1}{6}$ from the length) is now confirmed, as something spoken of explicitly.

B deals with the relation

$$x \cdot y + x + y = 1$$

in the same rectangle (and not, as claimed in the commentary in TMS, p. 67, with the same relation as A). The student is told to add 1 to both length and width and $1 \cdot 1$ ("turning" multiplication) to the accumulation, producing thereby the surface spanned by the extended length and the extended width. The addition of $1 \cdot 1$, and not of just 1, to the surface, and the exclusive use of the term "turning" in places where a "rectangular multiplication" is possible, indicates the geometrical interpretation shown in Figure 22B.

The description of the procedure is introduced and closed by the information that it is "Akkadian". This has been understood as an opposition to the teacher's own procedure. Since, however, the "Akkadian" procedure is also used in C, the real problem-solution, since the supposed "own" procedure of A was already met in a rather much older Akkadian text (AO 8862) from Southern Mesopotamia (cf. Goetze in MCT p. 148), and since, finally, this "Akkadian" procedure was not used in AO 8862, problem 2, where it would have constituted a great simplification, the interpretation of an opposition "Akkadian"/"Susian" is almost certainly false.

If opposition there is, it must rather be one between the mathematics of that southern area where the older texts, including AO 8862, belong, and the mathematics of the northern area "Akkad" where the younger texts contemporary with and terminologically similar to the Susa texts were made[36]. Yet, nothing in the formulation of the text enforces an opposition any more than the name "Horner's method" in a modern text suggests an opposition to other procedures for other problems. It seems most of all, that for the relation of B a procedure is proposed which bears a specific name, viz. "the Akkadian procedure".

Now, we notice that the procedure of B

is very close to the standard procedures for second-degree-problems: There, a gnomon is completed to a square, in B a quasi-gnomon is completed to a rectangle. Since the solution of second-degree problems is a characteristic of Akkadian mathematics from its early times (cf. BM 13901 and AO 8862, both discussed above), while Sumerian mathematical texts from earlier periods do not exhibit this feature[37], the term "Akkadian procedure" might simply be the name of the completion technique. Which more adequate name could have been chosen for a trick which, simple as it may look once it is found, was perhaps the starting-point for the whole fabulous development of "Babylonian" (i.e. Akkadian) mathematics; a trick which, when it was first found, will certainly have been noticed as a novelty?

C is a real mathematical <u>problem</u>:

$$x \cdot y + x + y = 1 \qquad (I)$$

$$\frac{1}{17}(3x + 4y) + y = 30' \qquad (II)$$

First, (II) is multiplied by 17 and thus transformed into

$$3x + 21y = 8^{\circ}30' \qquad (II')$$

with many pedagogical explanations like those of Texte XVI (see pp. 86ff), applying the "accounting explanation" of the coefficients, and seemingly referring to nothing but mental and verbal representations. Next (I) is transformed according to the "Akkadian" procedure,

$$X \cdot Y = 2 \;,\; X = x+1,\; Y = y+1 \qquad (I').$$

It will be noticed, that the extended length X

and the extended width Y are spoken of as "length" and "width" of "2 the surface". This is most natural in the geometrical interpretation. If, however, we think of the arithmetico-algebraic interpretation of the term as "the value of a second-degree polynomium in one or more variables" (cf. pp. 35f, and note 15), the possession of "lengths" and widths of some (but not all) such generalized surfaces is far from obvious, and the term becomes strange.

The two equations are then harmonized so as to constitute a standard problem,

$$3X \cdot 21Y = 3\cdot 21\cdot 2 = 2`6^{\circ} \qquad (I'')$$

$$3X + 21Y = 8^{\circ}30' + 3 + 21 = 32^{\circ}30' \qquad (II'')$$

which is, regrettably, solved as a standard problem needing no didactical explanations. Finally, 3X and 21Y are reduced (equally by uncommented algorithm) to x and y.

Before leaving these Susian lecture notes, I shall make three final comments on the presentation of the text in TMS.

First, all three sections of the texts are said to deal with the KI.GUB.GUB (KI.DU.DU, according to ABZ), which is claimed to be a technical term for the "additive fixed constant". In fact, the term only appears in A, while the same extension appears in B and C also, designated however in line 32 by the interpretable <u>waṣūbum</u>, "extension". The analogy with the well-known multiplicative "fixed constants" (IGI.GUB) implied by the term of TMS is unfounded, while of course the con-

If ki-gub-gub *itself should be interpreted, it migth be as "being placed firmly on the ground" and perhaps even "... on a socle" - not the worst possible description of the situation of Figure 22A.*

crete interpretation in terms of a change of variable is quite precise.

Next, Ú.UL is said to be equivalent to both kamārum, "to take together", and to kullum, taken to be the base of šutākulum, "to give reciprocally"; the two terms (of which one is only used additively and the other for certain multiplications only) are claimed to be interchangeable; and so, the term is translated first as an addition, next as a multiplication, and third not at all. In fact, in all its appearances, if the term were exchanged with šakānum, "to pose", customary constructions would be obtained. So, "pose" would seem an adequate translation. (Cf. p. 79).

The normal reading of Ú.UL ... Ú.UL ... Ú.UL as ūl... ūl ... ūl, "either ... or ... or" is not possible, because the outcome of line 3 is used in lines 4 and 5. So, unless the Susa-scribe used the Akkadian expression erroneously, as "first ... then ... then", an interpretation as a (pseudo-) Sumerogram is required.

Since Susa-"Sumerian" is characterized by its mistaken use of homophones (so, si instead of si₈ in ib-si₈), it is possible if not probable that Ú is a mistake for Ù, in which case it could be meant as a verbal prefix. However, the absence of such prefixes in other Sumerographic verbs speaks against this. Another possibility is that Ú.UL is a composite verb.

That such a verb might be conceptually related to "posing" is suggested by a cluster of established equivalences:

gar~*šakānum, "pose"*
gar-gar≈UL.GAR~*kamārum, "to take together"*
UL.UL *is, if not an ideographic equivalent of šutākulum at least semantically a close relative (cf. p. 64f, the marginal note), mainly used for the construction of squares.*

So, gar *is "posing". Continuing or repeated "posing" is collection into a heap ("taking together"), and so is the composite verb* UL-gar. UL.UL, *repeated, continuing or emphasized* UL, *places a line and "its equal" as a square frame.*

The claim that Ú.UL might be an equivalent of šakānum should not be taken to imply that šakānum have both additive and multiplicative meanings. It has none of them; but neither does the text require additive or multiplicative meanings of Ú.UL.

Finally, the problem (I″) - (II″) is of course of the well-known type x+y=A, xy=B, which was dealt with extensively in both MKT and TMS. It is no sensational proof (nor, of course, a disproof) of Thureau-Dangin's conjecture, that the Babylonians knew the double solution of the equation in one variable $ax^2+c=bx$ (an equation which they never formulated, as we may remember - see above, p. 48)*.

DEVELOPMENT AND LEGACY

The didactical expositions support the picture of Old Babylonian algebra which we had already drawn on the basis of other evidence. So we may conclude,

* In fact, the necessity to distinguish the two problem-types was already stated explicitly and clearly by Thureau-Dangin (1937a:16).

- that Old Babylonian algebra made extensively use of a geometrical heuristic for the solution of all mixed second-degree-problems;

- that for problems of the first degree, it would in some instances make use of the same geometrical intuitions as those known from second-degree problems, while in others it would base itself on predominantly or purely mental and verbal representations;

- that the techniques of the geometrical heuristic included cut-and-paste-procedures (or, otherwise expressed, the partition and rejoining of figures); the completion of figures by means of rectangles or squares, perhaps known as "the Akkadian procedure"; changes of scale in one direction, transforming rectangles into squares; the use of any adequate lines of the resulting figures as the basic "variables" of the standard procedures, as well as the representation of non-geometric quantities like rates of exchange by lines; and an "accounting technique" permitting the calculation of intuitively meaningful "coefficients";

- that mathematical reasoning would often if not always be made by reference to figures of known dimensions, a fact which underscores the extent to which the texts containing a description of the procedure must be read as expositions of a method (and which explains certain short-circuited procedures);

- and finally that the mathematical terminology, without being a stiffened technical vocabulary, reflects the procedures used, not least in its categorical structure, distinguishing sharply between various sorts of multiplications (multiplication of two abstract numbers; multiplication of anything by an integer; multiplication calculating geometrical quantities, e.g. by arguments of proportionality; and constructions of rectangular figures to which an area is implicitly ascribed) and at least two sorts of addition (accumulation and extension). Since the categorical distinctions must correspond to conceptual distinctions, we may say that Old Babylonian algebra was, deep into its conceptual structure, <u>a science about its means and methods</u>[38].

So, Old Babylonian algebra was a branch of mathematics very different from later mathematics - both from the socalled "geometric algebra" of Greek mathematics and from the arithmetical algebra of later times dealing with numbers and arithmetical operations. But Old Babylonian algebra is generally recognized to be the point of origin of at least the arithmetical algebra. So, at some moment between Hammurapi and al-Kwhârizmî, a change must have taken place.

The continuity from Old Babylonian to Seleucid mathematics is a well-established fact. Less well-known is the fundamental change

which appears to have taken place during the 1300 years separating the two periods.

A full discussion of the indications of this change would constitute a major investigation in itself. This I shall omit, and replace by some short remarks concerning the Seleucid text BM 34568, problem 9 (translated p. 9).

The first thing to strike the mind is that the procedure applied is not the usual one. 14 is not broken into halves but just squared. An eventual geometrical reconstruction of the procedure has to be the one of Figure 23, where $(x-y)^2$ comes about by means of some accounting or adequate cutting (see Figure 23B) when $4\cdot xy$ is subtracted from $(x+y)^2$.

This is already striking - the "Akkadian procedure" is replaced by something new and seemingly unfamiliar. However, taken in itself the observation has no necessary implications since, firstly, the same figure is already described in an Old Babylonian text (YBC 6504, problem 2, cf. Figure 13A; even for the incomplete problem 19 of the very early BM 13 901 it may have been employed), and since, secondly, that variation of the figure which is shown in Figure 13C would serve the solution of problem 10 of the same Seleucid tablet. The neighbourhood may have led to methodical off-set.

On the other hand, more decisive observations can be made. The initial multiplication 14·14 is designated by the term a-rá, "times", used in the multiplication tablets, where in Old Babylonian times a term for rectangular construction would be expected. 4·48, which in Old Babylonian would be described as a "doubling until four times" or by some similar expression, occurs as 48 GAM 4 - and the same term is used for all the following multiplications. Halving is expressed as a multiplication by 30′. No breaking into two, and no questions for a side occur - instead, the square-root is expressed as the solution to R GAM R (so, in this period, GAM seems to be nothing but an ideogram for a-rá). All conceptual distinctions between the different sorts of multiplicative operations have vanished, and everything is thought of in terms of the arithmetical multiplication. Algebra has become a science of unknown numbers - arithmetical algebra is already present.

Addendum: *In one of the Susa texts (TMS XXIV)* GAM *is simply used as a separation sign between numbers and "places". Could it be that this very simple sign had taken on the role of a genuine arithmetical symbol in the Seleucide period? (Without referring to a period, ABZ states that* GAM *can be used in the same function as the separation sign - no. 362 and no. 378, respectively; in the Seleucid table MM 86.11.410 and appear indiscriminately as a separation sign, cf. MCT p. 15). (In VAT 7848, a multiplicative* GAM *is written - MCT 141 n. 328a). Even the substractions are purely arithmetical, as reflected also in the translation on p. 3: You "go up" from 3`12˚ to 3`16˚, i.e. you* count *from 192 to 196 to find the difference (the term is* nim*).*

Will this mean that geometrical heuristcs was already forgotten by the third century B.C.?

Not necessarily, and probably not. Even if algebra was no longer a science of the procedures of geometrical heuristics, geometrical heuristics might very well be used as the basis of arguments, or as an illustration that the procedures applied were correct (as it is used by al-Khwârizmî). Direct evidence that the geometrical heuristics was still alive in the classical period might also be present in Diophant's Arithmetic.

The reference to Diophant may astonish. Obviously, Diophant does not argue by geometrical heuristics. His is an algebra dealing with numbers, rhetorical in its origin but so syncopated that one may wonder whether Tannery's translation (1893) using symbolic algebra, or Ver Eecke's (1926) expanding the formulations into full sentences is the more congenial.

On the other hand, there can be no doubt that Diophant knew the descendants of Babylonian algebra, and that Babylonian algebra was at least part of the foundation on which he made his building. Since already Neobabylonian algebra tended to be a science of numbers, this is no contradiction.

Precise knowledge of at least some features of Babylonian mathematics in late Antiquity or Byzantine times is implied e.g. scholion to Elements *X, def. 4, where the diagonal of a square with side 5 is given sexagesimally as 7˚4′15″50‴ (Vogel 1978:30 n. 16).*

The suggestion that geometrical support for algebraic reasoning was known to Diophant (or to some early commentator) is found in the Arithmetic, Book I, xxvii, xxviii and xxx. These three problems deal with equations which symbolically can be expressed

$x + y = a$	$x \cdot y = b$	(xxvii)
$x + y = a$	$x^2 + y^2 = b$	(xxviii)
$x - y = a$	$x \cdot y = b$	(xxx)

In all three cases, the condition which must be fulfilled if a solution shall exist is stated; and in all three cases a parenthetical remark follows, that "besides, this is figurative" (πλασματικόν - see Tannery 1893:I, 62-66, and Ver Eecke 1926:36-40, especially p. 36 note 5). The first condition is that $(\frac{a}{2})^2$-b be a square, the second that $2b-a^2$ be a square, and the third that $4b+a^2$ be a square. A πλάσμα.

It will be observed that the interpretation of the term given here (following ver Eecke) is an alternative to the one given by Sesiano (1982:192) as "constructible".

"image", "figure" in which the condition can be seen, is, for xxvii, that of Figure 9B; that of xxviii is seen on Figure 23A (in an application of that figure which is probably used in the Old Babylonian BM 13901, problem 19); and a πλάσμα of xxx is nothing but Figure 23B. Babylonian geometrical heuristics may here be spoken of by Diophant as a background since long left behind but not yet forgotten - or a commentator may have come to think of the possibilities of that other tradition. We may presume, then, that the geometric procedures described by the two 9th-century Islamic algebraists al-Khwârizmî and ibn Turk (see Sayili 1962: 162-169) are still descendants from Old Babylonian procedures and no independent development.

Addendum: *As Anbouba (1978: 76) points out, the geometrical method mentioned in the* ***addendum*** *on p. 72 is presented by al-Khwârizmî (or rather, not presented but implicitly used) in a way which clearly demonstrates its appurtenance to an established tradition well-known to him.*

This concerned the development of arithmetical algebra. But even in relation to the "geometric algebra", our results may put earlier discussions into a new perspective. It has been much debated, whether the presumed Pythagorean "application of areas" (cf. Heath 1921:I,150ff) and related problems and techniques were a translation into geometrical language of Babylonian arithmetical algebra - i.e., whether the term "geometric algebra" was at all legitimate. However, if Babylonian algebra was not, or not solely, an arithmetical algebra, the question must necessarily be put in another way: Was Greek so-called geometric algebra a deductive rationalization, a theoretical reconstruction (and, by the time of Euclid,

an axiomatization) of Babylonian geometrical heuristics, or was it, at least, a theoretical investigation inspired from the practice of geometrical heuristics, in the way so many parts of Greek science were theoretical investigations originally inspired from practice or every-day observations?

When formulated the latter way, the question may very well turn out to have an affirmative answer.

APPENDIX. ON BABYLONIAN GEOMETRY

If Babylonian algebra was, conceptually and methodologically, based on a naive geometry, it must be natural to ask for the connections between the geometry used in the algebra and the "real geometry" of the Babylonians. It would be puzzling and, indeed, a serious objection to the geometrical interpretation of the algebra, if the two were not closely related.

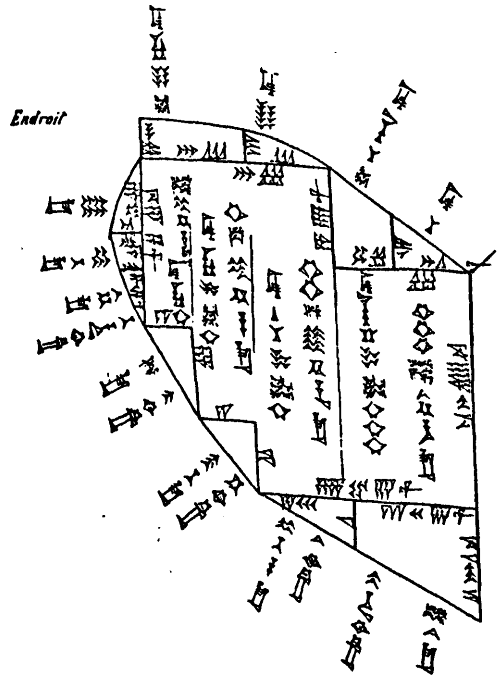
As a first approach, we may look at the surveying practice from which the Old Babylonian mathematical geometry must have grown (according e.g. to its vocabulary). A field plan from the earliest part of the second millenium B.C. will be a convenient starting-point (from Thureau-Dangin 1897).

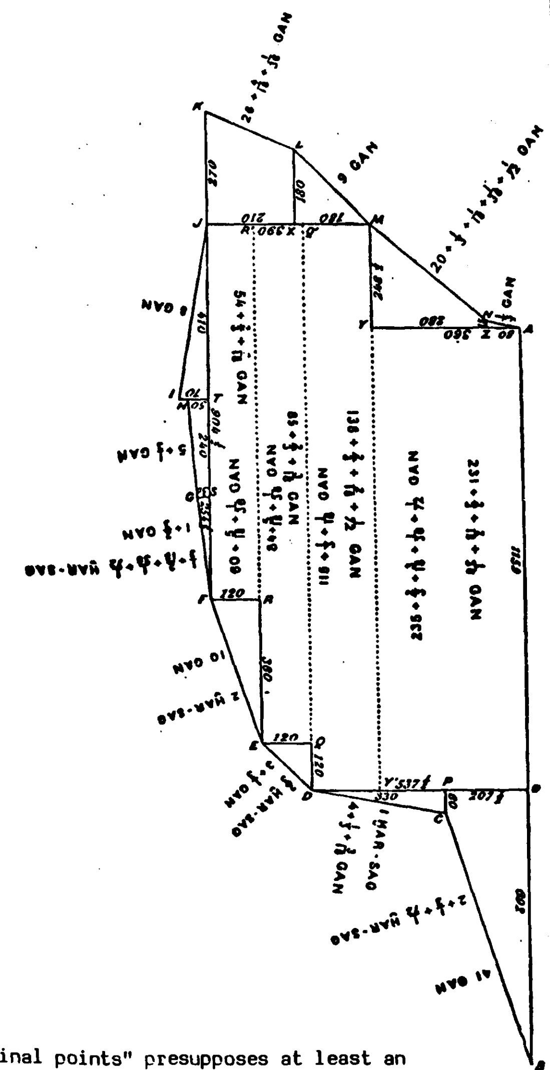
At the tablet, the plan looks as the figure to the left (next page). To the right, the plan is drawn in true proportions. The dotted lines to the right correspond to lines on the tablet which are very lightly drawn. They do not delimit fields or measured lines – they only enter as auxiliary lines for the calculation.

Several features of the plan and the calculation are worth noticing. First of all, that the total area is found as the sum of "good" areas: The surface is cut into a number of (approximate) rectangles (which together make up the šag̃ temen-na, the "inner field" – cf. ŠL II, no. 376.39), (approximately) right triangles and (approximately) right trapezoids*.

šag̃ temen must either mean the "inner temen" or "the interior (heart) of the temen" – where the sense of temen in this context is somewhat unclear. According to AHw p. 1346, the Akkadian loan-word temēnnum has to do with the juridical procedures concerning landed property ("Grundstein; Gründungsurkunde"). ŠL II, 367:39 gives "Erdaufschüttung, Fundament, Grundsteinurkunde" and hypothetically "Terrasse? Idealfigur?" in connection with the occurrence of šag̃ temen in another field plan.

* The approximate character of the right angles is emphasized in all cases in order to stress that there is no reason to think that the Babylonians had any general concept of angles as something to be measured. Still, they must have had a "practical concept" of rectangularity, permitting them to distinguish "right" from "wrong" angles, or "good" from "bad" angles, in drawings as well as in building and field-measuring. Even the expression "the four winds" in the sense of





Plan of the fields belonging to the district Šulgi-sib-kalama as rendered on the tablet MIO 1107 (left) and redrawn in true proportions. From Thureau-Dangin 1897:13, 15.

"the four cardinal points" presupposes at least an intuitive structuration of the plane by right angles.

It is impractical to carry on a constant reminder "(approximate)". So, in the following, whenever "right" angles are spoken of, they are to be understood as "good" or "not wrong", not as being 90° or the exact half of 180°.

The Egyptian parallel demonstrates that such a practical concept of rectangularity can be precise to a degree which is only limited by material tools. The Egyptians never built up a theoretical understanding of angles, and until the end of the Old Kingdom no general fractional symbol below 1/5 occurs (cf. Høyrup 1980:31 and note 94); and yet, the precision of the rectangular orientation of the Old Kingdom pyramids is justly legendary.

The area of the "inner field" is calculated twice, first "starting from the right", second "starting from the left", as

$$OA\cdot AY+(OA+YM)(OD-AY)+(OA+YM-DQ)\cdot QE$$
$$+(OA+YM-DQ-ER)\cdot \tfrac{1}{2}(RF+\{AY+MJ-OD-QE\})$$

and as

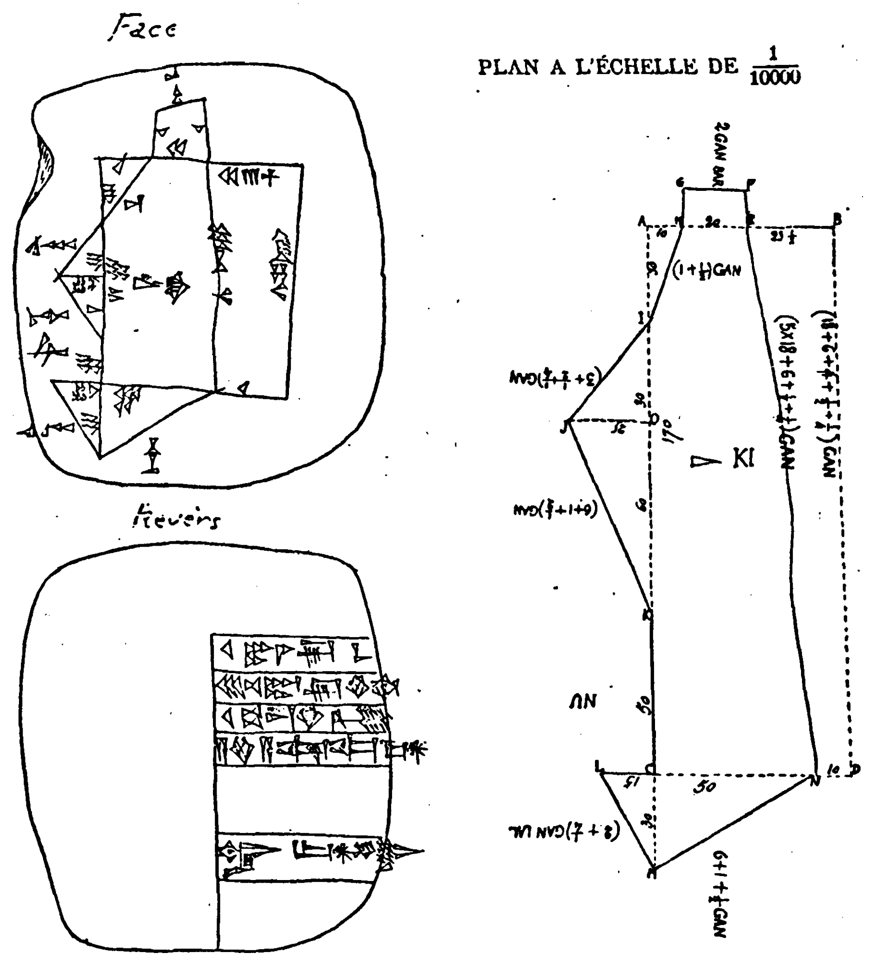
$$JF\cdot FR+(JF+RE)\cdot EQ+(JF+RE+QD)(JM-FR-EQ)$$
$$+(JF+RE+QD-MY)\cdot \tfrac{1}{2}(YA+\{FR+EQ+DQ-JM\}),$$

and the average is taken (partly perhaps because the measurements are not precise, partly because the right angles are only approximate, the two results differ by c. 3%).

A final remarkable feature is the drawing out of scale. The drawing cannot be understood as a naturalistic picture of the terrain; it is rather meant as a structural diagram, the carrier of a set of numbers, representing their mutual relations inside the total structure. Most strikingly this is seen in case of the lines TI and TH, of lengths 70 and 50 GAR, respectively, which on the original sketch are represented by one and the same line, with one of the lengths written above, the other below the line. The straight line representing the rather sharp angle DCB is another deviation from facts which must have been immediately visible to the surveyor.

Another field-plan slightly antedating the Old Babylonian period was published by Allotte de la Fuÿe (1915), whose autography and drawing in true proportions are shown on page 105.4. Once again the drawing is made without concern for true proportions, and once again the total area is calculated from simpler partial areas. This time, however, another feature of the "geometrical heuristic" turns up: The area of interest (fully drawn line on the drawing to the right) is not simply cut into partial areas. Instead, two completions are made: One, the triangle IAK,

in order to procure a more regular figure (a right-angled trapezoid), the other for less obvious reasons (probably reasons which should be sought in the physical or juridical* characteristics of the terrain and not in the geometry of the drawing).



Field plan with completions, as drawn on the tablet in free-hand sketch (left) and redrawn in true proportions. From Allotte de la Fuÿe 1915:49.

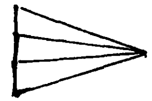
* Once again, the expression sag.-temen occurs for the relevant part of the central trapezoid.

All in all, the two field plans imply a number of conceptual habits close to those used in the geometrical heuristics of the algebra:

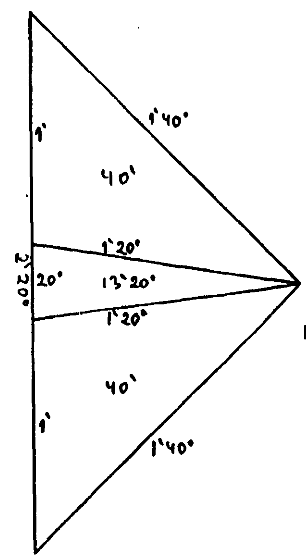
- Complicated surfaces may be cut into pieces which are simpler;
- furthermore, simplifications can be obtained by means of completions;
- the figures which are sought for in these partitions and completions are rectangles, right-angled triangles and right-angled trapezoids - where the right angles in question are probably not those of axiomatic geometry, rather such which correspond to an intuitive concept of the "correct" position of the lines which meet, but where, on the other hand, the drawings leave no doubt that an approximately right angle was looked for;
- the drawings made to represent the measured land have the character of structural diagrams, where the right angles are represented such as to be recognizable, but where the lengths of the lines implied are represented by numbers written along the lines, not by any attempt to make the drawing in right proportions (and where no care is taken to represent angles with no structural role correctly);
- finally, errors arising from the application of algorithms corresponding to "good" surfaces to less good surfaces are compensated for by averaging - between total areas of the system of rectangles in the first plan, between upper and lower width in the case of right trapezoids.

Theoretical extrapolations: YBC 8633

A geometrical text with no direct implications for the understanding of the algebra but important for the elucidation of the character of Babylonian geometrical conceptualization is YBC 8633 (MCT pp. 53-55). The text runs as follows:



The diagram as made on the tablet by impression of the stylus - cf. MCT Plate 29 (photograph) and 4 (autography). Inscribed numbers have been omitted.



Drawing showing the correct proportions of the triangle together with the numbers inscribed on the diagram in the tablet.

Obv. 1. A triangle (sag-dù). 1`40° each of the two lengths. 2`20° the width. The surface how much?
2. You, from 2`20° the width which ...
3. 20 tear off from the width of the triangle ...
4. and the 2` which you have left to two break: 1`.
5. 1` the width of the first triangle, 1`the width of the second triangle.
6. The second length how much?
7. 20 the makṣarum to 4 raise, 1`20°
8. 1`20° the true length (MCT's emendation "second length" is unnecessary) and 1` the width of the triangle;
9. (1`) to two break, 30 to 1`20° the [true] length
10. raise, 40` the surface of the [first] triangle.
11. 20 the width of triangle to two [break]
12. 10 to 1`2[0° the true length raise]
13. 13`20° the surface of [the second] triangle

Rev. 1. 1` the width of triangle to t[wo break],
2. 30 to 1`20° the true width [raise,]
3. 40` the surface of the thi[rd] triangle.
4. 1``33`20° the true surface ...

----

5. The makṣarum, that of tr[apezoid of diagonal(?)],
6. 20 to 5 , that of diagonal, raise,
7. 1`40° the diagonal it gives you.
8. 20 to 4 the length raise, 1`20° the true length.
9. 20 to 3 raise, 1 the width of triangle it gives you.

----

10. The makṣarum, that of trapezoid of diagonal,
11. 20, to 5 raise, 1`40° the length.
12. 20 to 4 raise, 1`20° the second length.
13. 20 to 3 raise, 1 the width of triangle.

The problem is to find the area of an isosceles triangle, the two "lengths" of which are obviously not considered sufficiently perpendicular to the base for the normal ½width-times-length-calculation to be adequate*. The procedure is inspired by the one used in surveying, where right-angled triangles are cut off from irregular figures. One could expect the height to be drawn, i.e. the triangle to be divided into two right-angled triangles. This is not done, however,- and for good reasons. Such a division would presuppose the height to be found, either by measurement - which would presuppose that the diagram made was a model made to scale, and which would furthermore fall outside the scope of a problem of <u>calculation</u> of an area - or by calculation - which would result in an irrational value for the height, and which would therefore imply the choice of a definite approximation. Also the latter possibility would fit badly into the normal pattern of Babylonian mathematical exercises (the application in the exercises of pre-established igi-gub-values is a different matter, be they exact or approximate as in the case of the circle).

Instead, <u>two</u> triangles are cut off which can (supposedly) be calculated exactly, and which are (supposedly) acceptably right. In other words, these triangles possess besides the length 1`40° another <u>true</u> length - true in the sense that it can be used to calculate the area as ½ width times length.

The two calculable triangles are found by

* Ironically, the two "lengths" are so close to being mutually perpendicular that half their product is only 0.02% off the true area! But this is far from obvious on the distorted drawing on the tablet.

means of a trick, the nature of which only becomes absolutely clear in the end of the text. It is used that 1`40° = 5·20, and so, inspired by the 3-4-5-triangle, the two outer triangles are supposed to have sides 3·20, 4·20 and 5·20. Provisionally, this rabbit is guarded in the teacher's sleeves, and instead of cutting off two times 1` (=3·20) from the width of the original triangle, he tears off 20°, leaving 2 times 1` which - miraculously, it seems - fit the 3-4-5-triangle with a <u>makṣarum</u> of 20 (a term to which I shall return below, but which can already here be seen to have the technical meaning of a factor of proportionality). The corresponding length of 1`20° is considered the "true" length of all three resulting triangles, and their areas are - separately - calculated and in the end added to give the total - "true" - surface. The result is 1``33`20°=5600, while the correct result would have been very close to 1``23`19°=4999, and the observation that the two lengths of 1`40° are almost perpendicular would have given 1``23`20°=5000.

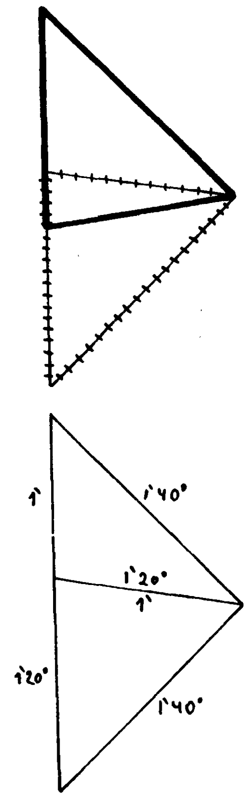
In the end, the rabbit is taken fully into the light. For each of the two outer triangles the sides are calculated as multiples of 5, 4 and 3, respectively.

Various observations can be made in connexion with the procedure, especially concerning the choice made between theoretical possibilities.

First we notice that the "true length" is treated as the <u>length</u> of three separate triangles, <u>not</u> as a height in the original complete triangle. Yet it would be computationally ~~just as simple to multiply 2`20° directly by~~ 1`20°, and it would require considerably less writing - and, arithmetically, the Babylonians knew very well that a·b+a·c can be calculated as a·(b+c). So, at least in a didactical text of the present type it must have been considered important or natural to stick to the idea that the area of a triangle is calculated as ½ width times (true) length (although, as we shall see, the concept of a height was perhaps present in the form of an ascending / descending line, a <u>muttarittum</u>). In surveying, where only right triangles were considered, this was of course a valid conceptual simplification (and only in few cases would it result in computational complication, cf. the field-plans on pp. 105.2 and 105.4 where only 2 non-right triangles are found). In "pure calculation"* like the present the same computational scheme was then applied as a fixed standard without necessary regard for the practical truth of the result obtained. The area had, so to speak, become a mathematical function <u>defined</u> by its appurtenant computational scheme, and its implications for just taxation or for the amount of grain necessary to sow a field was let out of sight.

This was, however, only a tendency, not a consciously chosen procedure as in modern abstract mathematics. Practical truth was not explicitly disregarded, it was simply

* The term "pure calculation" is of course coined in imitation of "pure mathematics", in order to point out a structurally similar relative autonomy from practical relevance and meaningfulness. On the other hand, the term "pure mathematics" is avoided as implying a substantial similarity which is not present. Only in a very general sense of the word was "mathematics" at all present in Babylonia,- in a general sense which covers practical computation as well as the relatively autonomous school exercises. Even the latter remained inside the range of calculation, and any confusion with Greek or post-Greek μαθήματα should be avoided. Cf. closer discussion in Høyrup 1983:9ff.



the object of <u>relative</u> neglect. This is seen in the interest of the present text to find a "true" length where the lengths present in the data of the problem are too obviously unfit for the normal computation. In cases, however, where the deviations from practical truth did not enforce themselves as obvious, the range accepted could be very wide - as in the case of bisections of a general quadrangle to which we shall return below.

A second observation is that the precision of the calculation could be considerably improved even on its own premisses. So, we might make an alternative calculation where the two quasi-right triangles overlap in the central area, having both a width of 1`20° and (according to the 3-4-5-scheme) a true length of 1`. Adding their areas and subtracting the overlap would give a total area of 1``10`=4200; the average between this value and the one obtained on the tablet would give 1``21`40°=4900. Or we might divide the width of the original triangle into portions of 1` and 1`20°, respectively; applying the same 3-4-5-rabbit we would get corresponding true lengths of 1`20° and 1` and a total area of 1``20` =4800.

Both methods would lead to considerable improvement of the precision of the calculation, and both agree with habits known from our field plans (including that of giving two different lengths to a line in a diagram). But both would presuppose that the calculations performed were looked at as explicit approximations to a transcendental true value (as would the calculation via an approximation to the irrational height of the original triangle). This was evidently not the aim of the text in question, or, in general, of the theoretical extrapolations from surveying into school geometry.In school teaching, the identity principle holds good, and the same length cannot be 1` and 1`20° at a time when you have no clearly expressible concept of both values as approximations; seemingly, the Babylonians had no such concept*, even though, as we have seen in the first field plan, they had practical understanding of approximations as well as of perpendicularity.

Two questions of terminology should be elucidated before we leave the text. One concerns the makṣarum, the other the "trapezoid of diagonal".

makṣarum is a derivation of the verb kaṣārum, "bind/bring/put together" (AHw I, 456). In non-technical contexts it means something binding or bound together - we may translate "bundle" or "bundling". In the mathematical text YBC 6295 (MCT p. 42) it is also found, seemingly applied differently - the text speaks of a method as a "bundling of a [cubic] root", "makṣarum ša ba-si", and tells that $\sqrt[3]{a}$ can be calculated as $\sqrt[3]{b}\cdot\sqrt[3]{a/b}$.

There is, however, a close methodological and conceptual affinity between the two mathematical occurrences of the term, and both can in fact be suggestively described as "bundling". In the present text, we look at the 1`-1`20°-1`40°-triangle as a multiplication by 20 of the 3-4-5-triangle - we "bundle" 3, 4 and 5, respectively, in bundles of 20. In YBC 6295, the problem is how to find the cube root of 3°22′30″=$3\frac{3}{8}$, a number

* Cf. also the fact that the Babylonians used methods which can be characterized as "single false position", in homogenous as well as non-homogenous linear problems (an example of the latter situation is given on p. 76), while the "double false position" belongs to much later times (see Vogel 1960:92f).

which does not occur in the table of cube roots. 7′30″, on the other hand, does occur. The tablet demands 7′30″ to be "posed" (in reality written) below 3°22′30″, and calculates 3°22′30″·(7′30″)$^{-1}$=27 ,- i.e., it calculates that 3°22′30″ is a bundle consisting of 27 times 7′30″. Finally, the cube root of 3°22′30″ is found as the cube root of 7′30″ times that of 27.

One may observe the close relationship between these "bundlings" and the discussions of BM 13901 N° 10 and BM 15285 N° 10 (pp. 22f). "Bundling" as spoken of explicitly in the rather different contexts of YBC 6295 and YBC 8633 seems to be a suggestive but still technical term for a standard technique occurring not only in these but also in many other texts*.

The other terminological question is the puzzling epithet of the bundling in rev. 10 and (in all probability) 5: "makṣarum ša sag-ki-gu₄ ṣiliptim", "bundling, that of sag-ki-gu₄ of ṣiliptum".

ṣiliptum derives from ṣalāpum "cut through/cross over (diagonally)" (AHw III, 1076b), and is used in mathematical texts both for diagonals of squares and trapezoids and for hypotenuses of (right) triangles. Since the outer length of our triangle is in fact considered a hypotenuse (and spoken of as a ṣiliptum in rev. 7), this need not disquiet us.

The more disturbing is the occurrence of a sag-ki-gu₄. Normally, this terms is used for trapezoids (so in VAT 7532 - see p. 51

* Incidentally, the numbers on the trapezoid of YBC 11126 (MCT p. 44 - see reproduction p. 22 of this paper) suggests a bundling of three. In spite of the translation offered, the occurrence in a molested Susa-text (TMS XVII, p. 95), finally, seems to suggest the same sort of factorization argument.

above) and Str. 367 (p. 73 above). It is, however, difficult to imagine what a trapezoid has to do in a context which with no doubt deals with a 3-4-5-triangle.

There seems to be two possible (and mutually exclusive) solutions to the puzzle, the first consisting in an extension of the meanings covered by the term sag-ki-gu₄, the other in a restriction.

The extension would imply the hypothesis that the meaning inherent in the term itself would be that of "quadrangle", and the further precision would depend on the context: If only a length and a width occur, a rectangle; if a length and an upper and a lower width, a trapezoid; if upper and lower length as well as upper and lower width, an irregular quadrangle. A sag-ki-gu₄ defined by having a diagonal would then by necessity be one with both diagonals equal, i.e. a rectangle. Since in any case the hypotenuse of a right triangle, e.g. the 3-4-5-triangle, is designated the "diagonal" of an implied rectangle, this explanation might seem plausible if only it had been supported by the slightest evidence.

It seems, however, to be unsupported. I have not found an instance where sag-ki-gu₄ means any sort of non-trapezoidal quadrangle. So, I am more inclined to believe that the term, apart that of a (right?) trapezoid in general, may have a more specific sense where a 3-4-5-corner occurs, and to which our text might refer. This assumption is indeed supported.

The key to this is an observation of an Akkadian loanword apparently derived from the term. Irrespective of its possible etymology as "forehead of an ox", the pronunciation of the first syllable must presumably have been sağ - and so, an identity with the Akkadian

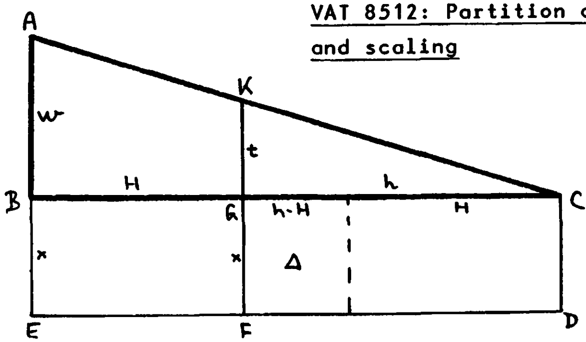
As a parallel it can be mentioned that the same coefficient list uses the expression uš ú sag, "length-and-width", to designate a specific rectangle (viz. the one with sides 3 and 4): 1°15′ igi-gub ša șilipta uš u sag (TMS III, 32). Evidently, the scribe is able to presuppose a concept of "the rectangle".

apsamiku(m) suggests itself* (as also seems to be claimed in AHw I,61a). However, several lists of coefficients (the Tell Harmal "compendium", cf. Goetze 1951, and the Susa list TMS III) as well as two problem texts from Susa (TMS XX and XXI) demonstrate that the apsamikum (written often as abusam(m)iku) can be referred to as a definite figure, presumably a specific trapezoid; furthermore, although the attempts to reconstruct the figure from the coefficients have led to no definite result, the values of the coefficients suggest that a 3-4-5-corner may be present (cf. Goetze 1951:138, TMS pp 110f, and Vajman 1959 (esp. p. 93).

Accordingly, I tend to advance the claim that the present sag-ki-gu₄ is the abusamiku of the coefficient lists, and not an undefined trapezoid.

* Cf. the equivalence between sag-dù and santakkum, "triangle".

VAT 8512: Partition of a triangle by completion and scaling



YBC 8633, interesting as it was for the elucidation of Babylonian geometrical thinking, had little bearing on and little relation to the methods of Babylonian algebra. Such relations are, however, easily found in the case of VAT 8512 (MKT I, 341f; TMB pp 101-103).

The text deals with a triangle (ABC on our figure) which is cut by a transversal GK in such a way that the difference Δ between the partial areas (ABGK-KGC) is 7`, and the difference between the heights (h-H) becomes 20. The width w is given as 30. H, h, the transversal t and (presumably) the partial areas are asked for.

In MLC 1950 (MCT, 48), the same number 20 is spoken of first as uš *, next as muttarittum* an-ta. *In the late Old Babylonian (or possibly Seleucid) text AO 17 264 (MKT I, 126f), muttarittum is used for both lengths of a trapezoid, lengths which cannot possibly both be perpendicular to the width.*
The latter text may have used the term as an unusual extrapolation form a normal "directly descending length" for the reason that the "lengths" in question are shorter than the widths.
All this does not ***prove*** *that the muttarittum of the present text is a length and no height. It is rather an indication that the word is no real technical term with a fully standardized meaning: Instead, it belongs to a tool kit of semi-technicalized terms which were used according to intuition to describe procedures and situations the conceptualizations of which were to some extent non-verbal. (Verbal conceptualization and full technicalization of vocabulary belong together).*

Obv.1. [A triangle, 30 the width. Inside two fi]eld parts*.
2. The upper surface over the lower surface 7 goes beyond.
3. The lower descendant** over the upper descendant
4. 20 goes beyond.

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* I follow von Soden's reading (1939:148) of the last word of the line as <u>ta-wi-ra-tum</u>, "Feldstücke, Teilflächen". The same concept should (in ideographic writing) be used for the partial fields in VAT 8389 and 8391 (pp. 76ff above) and Str 367 (pp. 73ff above), cf. Thureau-Dangin 1940:4.
The line count follows MKT, but with a typographical marking of those "continued lines" which are not counted separately in TMB.
Indications of restitutions of the text are omitted when subject to no doubt.

** The term "descendant" is chosen as translation of <u>muttarittum</u>, the "directly ascending/descending line" (cf. p. 105.9). In the figure, it is interpreted as a "true length", i.e. as a side in a right triangle. It would, however, make no difference except to the details of the argumentation if it were to be understood as a genuine height in a skew triangle.

5. The descendants and the bar* how much?
6. and the surfaces of the two field parts how much**?
7. You, 30 the width pose. 7` which the upper surface over the lower surface
8. goes beyond pose,
9. and 20 which the lower descendant over the upper descendant goes beyond pose.
10. The reciprocal of 20 which the lower descendant over the upper descendant goes beyond
11. undo: 3′ to 7` which the upper surface over the lower surface goes beyond
12. raise, 21 your head retain.
13. 21 to 30 the width append, 51
14. together with 51 cross†: 43`21°.
15. 21 which your head retains together with 21
16. cross: 7`21° to 43`21° append: 50`42°.
17. 50`42° to two break: 25`21°.
18. The side of 25`21° how much? 39.
19. From 39 21 which was crossed (<u>šakīltam</u>) tear off: 18.
20. 18 which you left is the bar.
21. Now, when 18 is the bar,
22. the descendants and the tw[o field parts how much]

For "undo" as a translation of paṭārum = du₈, cf. the marginal note above, p. 75.

An alternative interpretation of šakīltum would be a derivation from kullum, "which was retained". This would make the identification parallel to that in obv. 15, while the translation chosen is parallel to obv. 23. The choice follows from the structure of the text: In obv. 11-12, the number 21 is calculated by an inversed area calculation, (i.e. as GF). It has to be inserted at several places, and therefore to be remembered. In line 15, it is still referred to as the thing remembered, and here it is "crossed". By now, no further insertions have to be made, and so no further "retaining" is required. It will also be seen that the reference in obv. 23 is to the new identification obtained by the number in obv. 15. For both reasons it seems most probable that a reference to the number after obv. 15 should refer to the new identification made possible in this line.
Accordingly, šakīltum is to be read as a closely related variant of takīltum - cf. note 5a.

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* The term "bar" is chosen as translation for <u>pirkum</u>, which here denotes the line GK, and which in mathematical texts is normally translated "transversal". The word derives from <u>parākum</u>, "sich quer legen", "sperren", etc. (AHw II,828f); the word itself may, apart a line going through an area, mean the border of something (AHw II,855a) - the semantic common ground being probably something barring access or passage.
Since the technical meaning of "bar" at least in heraldic geometry coincides with that of <u>pirkum</u>, and the semantic connotations are also the same, I prefer it as the best mapping of the Babylonian concept at all its levels.

** Line 6 is translated from the restitution of TMB as emendated by von Soden (an emendation which removes Thureau-Dangin's main objection against his own proposal). Cf. p. 105.15 note (*).

† Here and in the following I will consider the "rectangulating" sense of <u>šutākulum</u> established, and therefore replace the deliberately empty "reciprocal giving" by the visually suggestive "crossing" - cf. Thureau-Dangin's translation "croiser".

23. You, 21 which you crossed to itself, from 51
24. tear off: 30 you leave. 30 which you left
25. to two break: 1[5 to 30 which you left rai
26. 7`30° that your head retain.

Edge 1. 18 the ba[r together with 18 cross:]
2. 5`24° [from 7`30° which your head retained]
3. tear off, 2`6° you leave.

Rev. 1. How much to 2`6° shall I pose
2. which 7` which the upper surface over the lower surface goes beyond
3. gives me?
4. 3°20′ pose, 3°20′ to 2`6° raise, 7`
5. it gives you.
6. 30 the width over 18 the bar how much goes beyond? 12 goes beyond.
7. 12 to 3°20′which is posed raise, 40.
8. 40 the upper descendant.
9. Now, when 40 is the upper descendant,
10. the upper surface how much? You, 30 the width
11. 18 the bar take together: 48 to two break: 24.
12. 24 to 40, the upper descendant raise, 16`.
13. 16` the upper surface. Now, when 16 the upper surface,
14. the lower descendant and the lower surface
14a. how much?
15. You, 40 the upper descendant to 20 which the lower
16. descendant over the upper descendant goes beyond
17. append: 1` the lower descendant.
18. 18 the bar to two break: 9
19. to 1` the lower descendant raise, 9`.
20. 9` the lower surface.

As already noticed by Gandz (1948:36f), the transversal (the "bar") can be and appears to be calculated by means of an adjoined rectangle BCDE, which transforms the triangle ACB into a trapezoid ACDE. The width x of the rectangle is chosen such that the prolonged transversal KF bisects the trapezoid - i.e.,

$x\cdot(h-H)=\Delta$, $x = \frac{\Delta}{h-H}$. Then the prolonged transversal x+t can be found from the relation valid for a bisecting transversal in a trapezoid

$$(x+t)^2 = \tfrac{1}{2}\{(x+w)^2+x^2\}$$

which was well-known to the Babylonians.

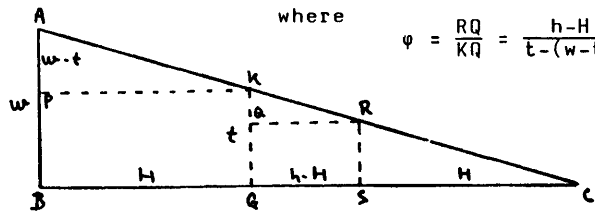
Indeed, the text starts by calculating x=21 as indicated, and then $x+t= \tfrac{1}{2}\sqrt{(x+w)^2+x^2}$ = 39, whence t=39-21=18. So, as first observed by Gandz and later e.g. by Vogel (1959:72), Vajman (1961:121) and van der Waerden (1975:73), the algorith employed yields a good support for the proposed adjunction.

Supplementary support follows from the formulations employed. Indeed, the addition in obv. 13 (waṣābum) and the subtraction in obv. 19 (nasāhum) are both "identity-conserving" as they should be. Furthermore, when later (obv. 23-24) the width w=30 is to be used, it is not referred to directly, although it was given and used before. Instead, it is found as the remainder when 21 (=x) is torn off from 51 (=AE). This could hardly happen if 51 were not an entity with its own independent mental or representational existence.

In passing we notice that the squarings of w+x (=AE) and x (=CD) are "rectangular multiplications", expressed by šutākulum.

From here, similar triangles could carry us easily to the end: $H = KP = AP\cdot\varphi = (w-t)\cdot\varphi$, where

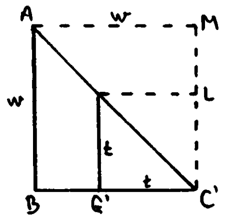
$$\varphi = \frac{RQ}{KQ} = \frac{h-H}{t-(w-t)} = \frac{h-H}{2t-w} = \frac{20}{36-30} = 3°20'.$$



In fact, this is almost what takes place. Yet φ is not calculated as the ratio between linear extensions, but instead as $\frac{\Delta}{\frac{1}{2}w\cdot w-t\cdot t}$.

This calls for an explanation. A comparison with the siege ramp will easily supply one.

It will be seen that φ is calculated as a ratio between areas. This reminds of the scaling in one dimension. The idea becomes clear if we draw the triangle as it would look without such scaling, as ABC': isosceles and right, i.e. inscribable in a square ABC'M.

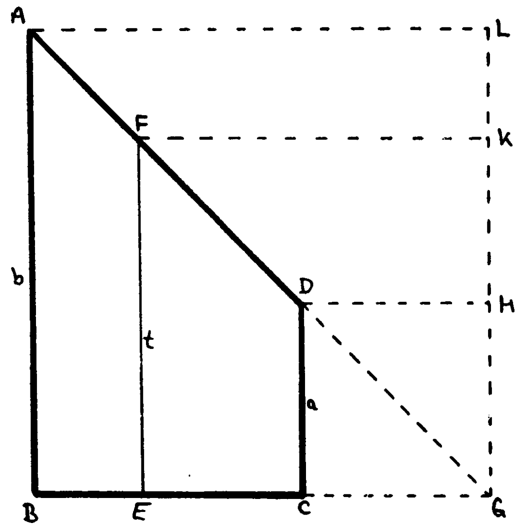


In this figure, the quantity $\frac{1}{2}w\cdot w-t\cdot t$ is meaningful. First we notice that $\frac{1}{2}w\cdot w$ is calculated by íl, "raising"*, as are areas of triangles and trapezoids (also in this text in rev. 12 and 19). So, $\frac{1}{2}w\cdot w$ is not the area of a rectangle with sides $\frac{1}{2}w$ and w but rather of a triangle ABC'. (Nor can it be half the area of the square, since this would be calculated by "crossing" and "breaking into two", as also observed by Vajman (1961:120). On the other hand, $t\cdot t$ is probably found by "crossing", and thus as the square K'G'C'L. The difference between the two is precisely the area by which the upper surface (ABG'K' = $\frac{1}{2}w^2-\frac{1}{2}t^2$) "goes beyond" the lower surface (K'G'C' = $\frac{1}{2}t^2$) <u>when no scaling is made</u>. So, the scaling factor φ is found in our text simply as the ratio between the real excedent Δ and the unscaled excedent Δ' (as already seen by Vajman - 1961:122). We notice that φ is calculated first, as a separate factor, which is afterwards applied to w-t (cf. above, p. 55, the marginal note).

When H = 40 has been found, the rest follows straightforwardly. The upper surface is found as $\{\frac{1}{2}(w+t)\}\cdot H$ ("taking together", "breaking", "raising"), h as H+(h-H) ("appending"), and the lower surface as $\frac{1}{2}t\cdot h$ ("raising")

* According to Neugebauer's restitution, which is accepted by Thureau-Dangin and von Soden, and which according to the photograph appears to be based on the space allowed in the line for the formulation of the multiplication. The restitution of a "crossing" in edge 1 appears to build on similar foundations.

With this in mind we may return to the initial part of the text, which provides us with a clue to the Babylonian derivation of the length of the bisecting transversal of a trapezoid. In obv. 13-16, the squares on x and w+x (the parallel sides of the trapezoid) were found by "crossing". So they must be understood as real geometrical squarings, not as arithmetical calculations following a standard formula once derived by guessing or by manipulations of general quadrangles (the areas of which were, as those of trapezoids, found by "raising"). If we look at the unscaled analogue of the triangle completed by the rectangle - i.e. at a trapezoid ABCD bisected by a transversal EF and inscribed along a diagonal AG in a square ABGL - the idea behind the calculation of the bisecting transversal ($2t^2 = a^2+b^2$) seems to be clear:



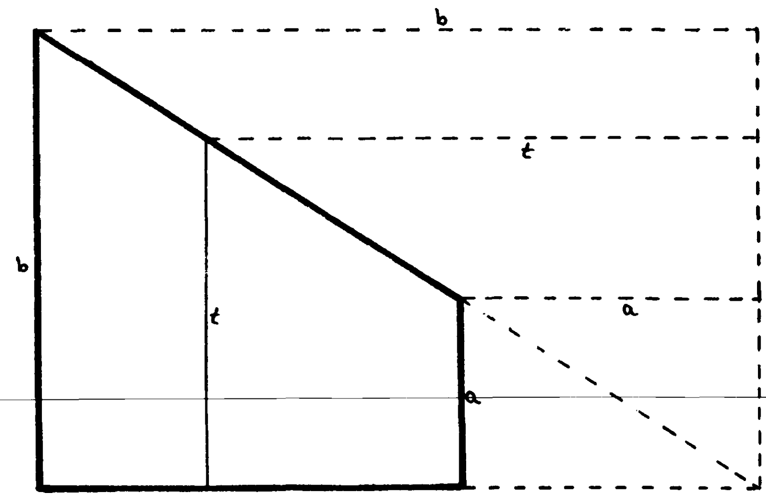
The square ABGL is bisected by the diagonal AG. In the lower half ABG a trapezoid ABCD is inscribed. This trapezoid is bisected by the transversal EF. Thus, the areas ABEF and CDFE are equal, and so are consequently the two gnomons ABEFKL and FECDHK. The area of the square on the transversal, which can be split into the area of the inner gnomon and the area of the square on a, is therefore also equal to the area of the outer gnomon and an excess equal to a^2 - and the double of the area of the squared transversal must thus equal the area b^2 of the total square ABGL, plus the excess equal to the area of CGHD: $2t^2 = a^2+b^2 = \frac{1}{2}(a^2+b^2)$. This is not only the formula we should have, and a formula the contents of which was known by the Babylonians; besides, our argument implies the squarings to be real geometrical squarings, as they should be according to the vocabulary used in obv. 13-15 *.

By appropriate scaling, the relation is immediately seen to hold for any trapezoid inscribed along the diagonal in a rectangle - and if we believe the above interpretation of the scaling factor φ, this would be obvious to the Babylonians. By further extension (one edge of the trapezoid sliding along the other) the relation can of course be seen to hold for any trapezoid - but whether that was obvious to the Babylonians, and whether they had any interest in such general trapezoids is more doubtful; as we shall see in the case of YBC 4675 below, they appear to have transferred the basic argument without further thought for the necessity of justification of the generalization to general quadrangles where it is not correct.

We may conclude that the Babylonian knowledge of the bisecting transversal in right-angled trapezoids follows easily by the standard methods of the geometrical heuristics, and that the vocabulary of our text suggests that this was exactly the way the argument was made by the Babylonians (and furthermore, that an actual argument was made when the single problem was treated, i.e., that the Babylonians did not make use of a standardized arithmetical algorithm - further evidence in the same direction follows below in connection with YBC 4675).

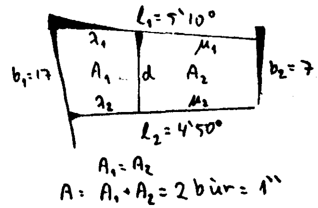
* For reasons of mathematical simplicity, Olaf Schmidt has already proposed that the Babylonians derived their knowledge of the bisecting transversal in the simple case of a trapezoid inscribable in a square and extrapolated from there (unpublished - oral communication).

The text, however, does more than establish a connection between the geometry of trapezoid bisection and the geometrical heuristics of the algebra. We observe that obv. 13-16 goes directly from knowledge of the parallel sides of the trapezoid to the crossings. Not the slightest hint of any intermediate operation or argument is to be found. This supports the idea that the unscaled "square trapezoid" is not regarded separately, as a new figure. Instead, we can presume, the inscription of the rectangular trapezoid in a rectangle is thought of as a diagram where lengths and widths can be imagined equal as well as unequal at will and after convenience - and of course, even though there is no doubt that this problem deals with a real geometrical figure it remains undecided whether any diagram was physically drawn or the scribe did everything by "mental geometry", just as in the case of the algebra.



Finally, the absence of a drawing in this indisputably geometrical text (and several others) shows that either another medium was used for drawings supporting the argument, or mental geometry was possible to an extent which would suffice for the algebraical problems.

Further theoretical extrapolations: YBC 4675



The diagram as given on the tablet. Symbolic designations follow MCT.

Another text dealing with a partition problem is YBC 4675 (MCT p. 44f). This time, an irregular quadrangle is bisected by a transversal, and a diagram is impressed on the tablet showing the structure of the problem as seen by the author - but definitely not the geometrical proportions correspond-ind to the numerical data concerning the sides, which, firstly, are grossly misrepresented and which, secondly, do not determine the figure (cf. the drawing showing two possible materializations of the numbers at the bottom of this page).

The text runs as follows:

Obv. 1. When a surface is spanned by length and length*, the 1st length is 5`10°, the 2nd length is 4`50°,

2. the upper width is 17, the lower width is 7, the surface is 2 bùr,

3. At 1 bùr each, the surface is divided in two. To what does the middle bar correspond?

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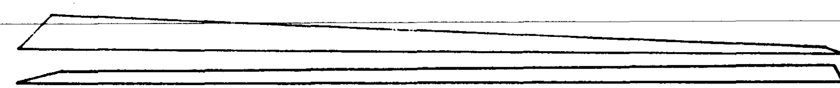
* The first part of the line is transliterated in MCT as "šum-ma a(?)-šà(?) uš uš i kú". Since kú, "eat", is used ideographically for both akālum and šutākulum ("eat" and "cross", i.e. "make eat one another"), i kú should probably be read as īkul, īkulū, ikkal or ikkalū ("it/they eat/have eaten); the double subject uš uš supports the plural interpretation īkulū/ikkalū, and the G-stem-construction indicates that the "eating" involved in "crossing" can be understood as the occupation of a space - as also proposed by Gandz (1939:417f) in his explanation of the use of ukullûm, "Verpflegungsration", as a term for inverse slope: The amount of horizontal extension occupied, "eaten", per unit of height. (As shown by Gandz, the use of "eat" in the sense of "occupy space" is current in the Mishnah and Talmud). The term "span" seems to be an adequate translation, especially in this case where the surface is spanned by two opposite sides, not as a rectangle by length and width; maybe it would, in general, be the translation to prefer.

Closely related to Gandz' Hebrew examples is the use of šakīltum in VAT 8389 obv. II,4, where the word refers to the še lul, the "false grain", i.e. the amount of grain "covering" a unit area (cf. above, p. 76).



Two possible materializations of the data of the figure

4. How great (kī maṣi) shall I pose the long length and the short length

"Arrive" translates sanāqum, "be/arrive to be close (to something)" (with connotations and probable etymology in the nature of "pressing"). So, a better translation might be "come close".

5. so that 1 bùr arrives; and to 1, second, bùr,
6. how great shall I pose the long length and how great the short length
7. so that 1 bùr arrives. Both[2] different complete
8. lengths[1]* you take together, into two halves you break.

"Come up" translates elûm, "be/become high", "go up"

9. 5` comes† up for you. The reciprocal of 5` which comes up for you you undo,
10. to the upper width which goes 10 beyond the lower width,
11. to 10, the going-beyond, you raise, 2´ it gives you.

[11a *(which, multiplied by the total surface A = 1`` gives 2`).*]

12. You turn around (tasahhar). 17 the upper width you cross:
13. 4`49° comes up for you. From inside 4`49°
14. 2` you cut off: 2`49° what remains (ahertu, from ahārum, "remain behind").
15. You take its side:
16. 13, the middle bar, comes up for you.
17. 13 the middle bar which comes up for you
18. and 17 the upper width you take together, to two halves you break:
19. 15 comes up for you. The reciprocal of 15 you undo,
20. to 1 bùr, the surface, you raise:

NB: In this line, the interchangeability of nadānum and elûm (when a result "is given" or "comes up") is seen. Instead of nadānum, the causative Š-stem of elûm could have been used, or instead of elûm the N-stem of nadānum.

← Rev. 1. 2` it gives you. 2` which comes up for you
2. to 2´ the arakarûm* you raise:

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* "Both different complete lengths" translates "uš-ha gamerūtim kilallēn". Since the dual is indicated both by the plural form of gamerum, "complete", and by kilallān, "both", the suffix -ha is probably more than a plural indicator (which anyhow is normally left out in the Old Babylonian mathematical "Sumerian"). Presumably it serves to emphasize that different lengths are involved - cf. Falkenstein 1959, § 18d.

† The singular form of the verb here and in corresponding constructions further on in the text shows that a number which results from a calculation is thought of as one entity which is describable by a measuring number, not as a collection of units. Cf. other texts above.

* "Factor"? Derivation from a-rá? In any case calculated as $\frac{\ell_1-\ell_2}{\ell_1+\ell_2}$ or, rather, as $\frac{(\ell_1-\ell_2)/2}{(\ell_1+\ell_2)/2}$. Cf. MCT p. 15.

Cf. marginal note p. 105.29

3. 4 comes up for you. 4 which comes up for you
4. to 2`, the length, you append: 2`4°, the long length;
5. 4 from 2`, in the second place*, you cut off:
6. 1`56°, the short length. You make: 1 bùr arrives.
7. You turn around. 13 the middle bar
8. which comes up for you, and 7 the lower width you take together,
9. into two halves you break: 10 comes up for you.
10. The reciprocal of 10 you undo: To 1 bùr the surface you raise:
11. 3` the length comes up for you. 3` the length which comes up for you
12. to 2´ the arakarûm you raise:
13. 6 comes up for you. 6 to 3` the length you append:
14. 3`6° the long length. 6 from 3` the length you cut off:
15. 2`54° the short length. You make them span**:
16. 1 bùr arrives.

Before entering the analysis of the text we notice that the total area of the quadrangle (2 bùr = 1``) must be determined as average length times average width, $A = \frac{\ell_1+\ell_2}{2}\cdot\frac{b_1+b_2}{2}$ = 5`·12 = 1``. If the author of our text had only tried to make a drawing in the style of those at bottom of p. 105.24, he would inevitably have discovered that the (average) "width" entering this calculation is very far from being a "true width", applicable for the calculation of practically true areas. So, he

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ki-X is very rare as an ordinal specification in the mathematical texts - it is only found in the Strassburg-texts in MKT I, in BM 13 901 (MKT III) and in Plimpton 322 and the present text in MCT. In all these cases, it can be interpreted as "in the X'th place", and its use is never parallel to the use proposed for the present occurrence in MCT.

* "In the second place" translates ki-2, which indeed it translates literally (ki, "earth", "place", is used as a determinative for names of locations). There is no reason in our text to interprete the term adjectively, as "second", as done in MCT. If this sense had been intended, then it would have been possible and natural to use šanum or 2-kam (or an išten ... išten-construction). Cf. also the use of the sign (ligated as ki-min) as a repetition-sign e.g. in vocabularies.

** "You make them span" translates tuštakkal, "you make eat one another", which was otherwise rendered "you cross" (and earlier "you give reciprocally"). The translation is made in accordance with the interpretation of obv. 1, cf. p. 105.24 note (**).

can hardly have cared about this. Instead, he drew a diagram on the tablet illustrating the mutual relations between the imagined lengths and widths of his quadrangle (which were certainly imagined so as to permit a legitimate solution to the problem together with a beautiful total area*) and used a calculational scheme yielding practically acceptable results when applied in surveying. Tendentially, the area had become a mathematical function defined through its computational scheme - cf. analogous arguments in the case of the triangle on p. 105.9.

Had the quadrangle represented a real field, the case would of course have been different. The deviation between "true width" and average width had been evident, and right triangles would have been cut off in the two ends of the field, leaving a right trapezoid. In that case, however, the dimensions measured and used in the calculations had been the "true" lengths and widths of the partial areas, and no surveyor would probably even try to calculate an alternative area according to the method used in our text. So, no palpable paradox would arise which could undermine the credibility of the concept and calculational schemes of areas.

Nor did such paradoxes arise from the use of the theoretically extrapolated area concept in texts like the present. As shown by Bruins and Peter Damerow** in somewhat different

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* $\frac{1}{2}(b_1^2+b_2^2)^2$ is itself a square, as it should be (cf. below) and the total area as calculated by the "surveyors formula" is 2 bùr = 1‵. Finally, the deviations of the two lengths from the average length yield a regular and simple value for the arakarûm. If simplicity was intended, very little choice was left open concerning the dimensions of the figure.

** TMS pp. 4-7; Peter Damerow, "Anmerkungen zum Text YBC 4675". Working paper for the Berlin workshop, August 1983.

ways, the "surveyors formula" yields an additive area when a quadrangle is bisected by a transversal calculated as done in our text (cf. below). So, even here the Babylonians could avoid shattering discoveries thanks to harmony between the problems investigated and the methods applied. Calculational schemes arising by extrapolation and used as basis for exercises in pure calculation could survive because they flourished in a protected ecological niche.

Asaḫḫir ("I turn around") - namely to the structure of the text. It can be separated into a number of main sections:

a: Statement of the problem. Obv. 1-7.

b: The "middle bar" (which bisects the "field" in a way analogous to that of a parallel transversal in a trapezoid, dividing the opposing sides proportionally) is calculated to be d=13. Obv. 7-16.

c. The (average) length $\frac{\lambda_1+\lambda_2}{2}$ is found for the upper left field. Obv. 17-rev.1.

d. λ_1 and λ_2 are found as $\frac{\lambda_1+\lambda_2}{2} \pm \alpha\frac{\lambda_1+\lambda_2}{2}$, where α is the arakarûm - cf. p. 105.25 note (*). Rev. 1-6.

e. Like c, for the lower (right) field. Rev. 7-11.

f. Like d, for the lower field. Rev. 11-16.

Of these sections, the least transparent is b, which I shall therefore leave aside for a moment, jumping into the text at obv. 16-17, at a point where the middle bar d is supposed to be known. For the upper (i.e. left) field the area A_1 and the opposing widths b_1 and d are then known. The average width is found to be 15 (but given no explicit name), and the average length (spoken of simply as length, uš) is found, evidently from the supposition that the area is the product of (average) length and (average) width. Even though the average width is given no name, the use of the unqualified term uš leaves little doubt that approximately so were the words of the implicit argument.

This is an important clue to the area concept of the Babylonians. It shows us, that not only was the area formula which was applied for general quadrangles obtained as a generalization from rectangles and/or right trapezoids; it was also still spoken of in terms which point back to the rectangle, not as a mere scheme for manipulation with numbers*. To state the matter pointedly, the Babylonians were perhaps only in possession of one "formula" for the calculation of an area (which in this case would tend to constitute the contents of the area concept), "length" raised to "width", and of relatively coherent methods to extend the application of this formula to cases where it did not apply straightaway: Right triangles; right trapezoids;- and in "pure calculation" (cf. p. 105.9, note (*)) skew triangles and general quadrangles.

Possibly the terms uš-gíd-da and uš-lugúd-da should not be translated as "long" and "short" length but rather as "prolonged" and "shortened" length. This would fit the Akkadian equivalents (arākum / karûm) just as well (when D-stems are taken) and the procedure better.

It should be reflected whether the term arakarûm could possibly derive from sort of pun, viz. from a compression of arākum and karûm into one word, a "long-shorten" - perhaps made in analogy with some composite Sumerian verb. (According to Mogens Trolle Larsen, this can hardly be the case).

Section d calculates the real ("long" and "short") lengths of the upper field on the condition that their ratio (to each other or to their common average) be like that of the "complete" lengths. This is done via the arakarûm, the fraction by which these deviate from their average. The calculation of this entity is omitted for reasons which are not clear. In any case, however, the omission has a parallel in the omission of line 11a where 2′ are transformed into 2\`; similarly, the scribe may have overlooked that the 2′ $\left(= \frac{b_1 - b_2}{(\ell_1 + \ell_2)/2}\right)$ of obv. 7-11 is only numerically coincident

* All the more important is it then that the area of an irregular quadrangle (as well as that of a right trapezoid and a right triangle) was calculated by raising. "Crossing" cannot then be a term for the operation "to calculate the area of a rectangle" - and the constructional implications of the word ("spanning", "building a surface") are thus brought to the fore.

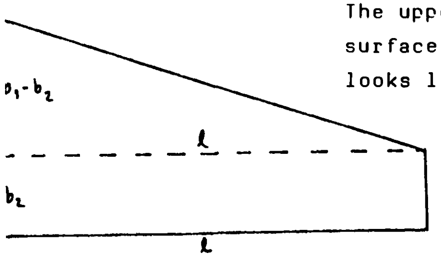
with the arakarûm.

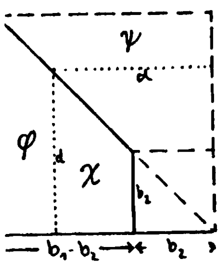
One might ask why the Babylonians chose precisely the solution with proportionally divided sides. Of course an answer could be that this requirement was inherent in the concept of a "middle bar"; on the other hand, this answer would only replace the problem by another equivalent question: Why a concept with such a specific content was used.

A counterquestion could be, whether the Babylonians bothered to find a complete set of solutions and, more fundamentally, whether they were aware that the problem might have several solutions. They may have been well satisfied with one standard way to solve the problem. And in fact, the proportional division works well: In simple trapezoids it is easily argued for (cf. p. 105.20 ff), and it is perhaps the only bisection which can at all be argued for on Babylonian cognitive ground; if used in practice for the division of fields it cannot (concave quadrangles disregarded) produce results which are more pathological (i.e. skew) than the initial quadrangle; and, more important perhaps, if the irregular quadrangle was thought of in analogy of the more regular right trapezoid (which it probably was, cf. below), the proportional partition would be naturally carried over to the general case. In an alternative formulation: If the concept of a "middle bar" was originally that of a parallel transversal in a right trapezoid (rectangularity as well as parallelty still of course to be understood as practical, not theoretical concepts), proportional partition would belong to it naturally even when carried over to other cases, unless the emergence of contradictions would enforce a conceptual restructuration - and no such contradictions would arise in this case.

Sections e-f are (apart the <u>tuštakkal</u> of rev. 15) completely analogous to sections c-d; so they call for no commentary. Instead, I shall turn to section b, the calculation of the transversal. It is worth noticing that the result found, $d^2 = 169 = \frac{1}{2}(b_1^2+b_2^2)$, is precisely what would follow from the "formula" normally applied to bisection problems - cf. also pp. 105.20 ff. As this calculation could easily serve in the contxt of this problem <u>but is not used</u>, it cannot be the argument behind our text - its idea cannot simply be a mechanical transfer of a calculational scheme known from right trapezoids.

What <u>is</u> then the idea? The text contains no direct explanation but still some clues: The upper width is "crossed", and the total surface is multiplied by something which looks like a scaling factor*.





In order to understand the clues we should observe that this factor is the ratio between the difference b_1-b_2 between the widths and the average length $(\ell_1+\ell_2)/2$. This suggests that the problem was thought of in terms of a figure having only one length - a length which does not need to be a common length but should rather be the <u>true</u> length of the figure, in agreement with normal Babylonian mathematical habits. This brings us back to the right trapezoid (or, equivalently, to the right triangle with appended

* The latter assertion presupposes that the restitution of line 11b is correct. There can, however, be no doubt as to this. The multiplication by a factor 1` is required by the orders of magnitude, and a multiplication by an area is required by dimensions. Nothing but A = 2 bùr = 1` fulfills both conditions..

rectangle known from VAT 8512 (pp. 105.15ff). If we replace the average length $(\ell_1+\ell_2)/2$ by the true length ℓ of this figure, the factor $\frac{b_1-b_2}{(\ell_1+\ell_2)/2}$ is replaced by $\frac{b_1-b_2}{\ell}$, which is precisely the scaling factor which transforms the upper, right trapezoid into the lower, "square" trapezoid".

As we saw above (pp. 105.20 ff), this figure would permit us to argue for the relation $2d^2 = b_1^2+b_2^2$, if d is a bisecting transversal. This is not what is done here; however, the procedure of the present text is easily explained on the same figure. The quantity $S = \frac{b_1-b_2}{\ell}\cdot A$ is simply the area of the full-drawn trapezoid, i.e. $\varphi+\chi$. Now, since d bisects the figure, $\chi=\varphi$, and for reasons of symmetry, $\varphi=\phi$. So, $S = \varphi+\phi$, and so $b_1^2-S = d^2$, exactly as stated by our text.

The interpretation is supported by the terminology employed. S is calculated by "raising", while b_1^2 is the result of a "crossing" - i.e., b_1^2 should be thought of as a real geometrical square. Finally, d^2 is found by "cutting off" S, i.e. by removing a part of b_1^2 equal to S, namely the outer gnomon $\varphi+\phi$.

Even if this is different from the method proposed above in the interpretation of VAT 8512, the kinship is hard to overlook. Further, the relationship with the interpretation of the siege-ramp-problems (pp. 53-63) is obvious*. Already the formulations ot these, pointing to different arguments based on a common figure, suggested that heuristic arguments rather than standard formulae/algorithms were applied. If we think of the relation between VAT 8512 and the present text, and especially

* A special affinity between the present text and BM 85210, obv. II.15-27 (see p. 56) will be observed.

of the striking non-application of the relation $2d^2 = b_1^2+b_2^2$, the same observation can be made. If finally we notice that "crossing" is, in each and everyone of these problems, used for the squaring of b's (or h's, their analogues in the siege-ramp-problems) and only there, we have, all evidence for the geometrical heuristic disregarded, good reasons to believe that the methods of all these texts were related,- probably in a way close to the one here proposed.

As a final important point concerning this text I shall remind of the occurrence of the terms kú and šutākulum in obv. 1 and rev. 15, at places which have clearly no multiplicative meaning, but instead, almost as clearly, a constructive sense. This was already taken note of on p. 105.24, note (*), and p. 105.26, note (**). Worth noticing is also the parallel character of rev. 6 ("You make: 1 bùr arrives" - teppešma 1(bùr)^{iku} saniq") and rev. 15f ("You make them span: 1 bùr arrives" - tuštakkalma 1(bùr)^{iku} saniq). There is no doubt left that epēšum, "to make", and šutākulum, "to cross", "to make eat each other" (and in the translation of rev. 15 "to make them span") are used here for the same process - which, in the case of epēšum can only be that of locating a field or a geometrical figure physically. (It will also be observed that dù may be used ideographically for both epēšum and banûm).

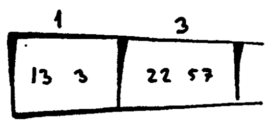
The stative form of sanāqum. "to arrive" (etc.) might perhaps be rendered more precisely as "1 bùr will be at hand" (the use of the stative is analogous to that of many omen texts).

Str. 367: A Trapezoid problem "of the first degree"

The trapezoid partition Str. 367 (MKT I,259f) was already presented in the main text (pp. 73ff). As stated in a marginal note, however, the treatment in that place turns out to be insufficient. I shall therefore give a full translation of the text followed by a new commentary.

The text is unusual by its language. Firstly, it contains very little syllabic Akkadian - only the words u, ša, ana, and ina; only the first problem of Str. 364 (MKT I,248f) is comparable, the completely stenographic series texts disregarded. Secondly, the Sumerian of the text is, if far from fully grammatic, supplied with so many (Sumerian) grammatical complements that it can hardly be regarded as logographic Akkadian. At the same time, the phrasing corresponds closely to the texts written in normal syllabic+logographic Akkadian.

The text runs as follows:



The diagram counted in MKT as obv. 1, drawn after the photograph in MKT II,Pl.12 (the autograph in Frank 1928 is very imprecise).

Obv. 1. [The diagram shown left.]
2. A trapezoid. In the inside 2 field parts*. 13`3° the upper surface.
3. 22`57° surface (N°) 2. The 3d part** of the lower length in
4. the upper length. That which the upper width goes beyond the bar,
5. and the bar goes beyond the lower width,

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* Cf. above, p. 105.15 note (*).

** "The 3d part" translates "igi 3 gál", a phrase elsewhere rendered as "the reciprocal of 3". Since in the present case the meaning is indubitably "the third part" of another quantity, our text provides us with evidence for the way in which the Babylonians thought of their "reciprocals": Not abstractly, as solutions to an equation $n \cdot x=1$, but simply as "the n'th part of 1". So was, by the way, already argued by Gandz (1936), on the evidence of tables of generalized "reciprocals" with respect to 10 and 70 instead of 1 (or 1`, as it is suggested by a tablet published by Scheil (1915). (Cf. p. 105.35 note (*)).

taken together, 36.

6. Their lengths, the widths and the bar what?
7. You, when you shall make, 1 and 3 may you pose.
8. 1 and 3 taken together, 4. The reciprocal of 4* undone: 15′.
9. 15′ to 36 lifted, 9 it gives. 9 to
10. 1 lifted, 9 it gives. 9 to 3 lifted, 27.
11. 9 that which the upper width over the bar goes beyond.
12. 27 that which the bar over the lower width goes beyond.
13. The reciprocal of 1 undone, 1 to 13`3° lifted,
14. 13`3° it gives. The reciprocal of 3 undone, 20′ to
15. 22`57° lifted, 7`39° it gives.

Rev. 1. 13`3° over 7`39° what goes beyond?
2. 5`24° goes beyond. 1 and 3 taken together, 4.
3. The ½ of 4 broken off, 2. The reciprocal of 2 undone, 30′ to 5`24°;
4. 2`42 it gives:. the false NU**. 2`42° is not undone.

The same distinction is expressed in VAT 7532 and VAT 7535 (MKT I,294f and 303ff). In these texts, the distinction is perhaps even more striking, because the complete phrase is also used when "the n'th part of 1", "1" taken as a representative of an unknown magnitude, is spoken of. (Cf. discussion of VAT 7532 above, pp. 52f).

* The text seems to distinguish "the n'th part" in the general sense and the specific technical sense as a number, viz. 1/n (translated here "the reciprocal of n") - while the former was expressed "igi n gál", the latter is written simply "igi n". So, even if Gandz (1936) was, according to our text, right when analyzing the Babylonian concept of a reciprocal, he was seemingly not right when suggesting implicitly that Neugebauer's translation ("the reciprocal ...") should be abandoned. When a conceptual distinction can be seen in the texts, it should not be blurred by a translation, be it made in order to avoid a term loaded with anachronistic temptations.

** The passage as interpreted in MKT reads "2,42 in<-sum> -ma nu lul". I translated as if -sum has simply been left out by error. If, as Neugebauer suggests as a possibility, -ma is a copying error for -sum, the ":" should change into "," in the translation.

Thureau-Dangin (TMB p. 90f) makes an Akkadian interpretation "2,42 in ma-nu lul" which he finds inexplicable. In principle ma-mu could be manûm, "count, calculate", or manûm, "mina", and the phrase would (with in extended into in-sum) mean "2`42° it gives, the false counting/calculation/mina". Two arguments, however, speaks against this. Firstly, the sudden occurrence of a specialized Akkadian term in a text containing only particles from this language; secondly, the linking of an abstract or a metaphorical concept ("counting", "calculation", a weight unit) with the term lul, "false", which otherwise serves to give concrete

5. What to 2`42° may you pose which gives 9?
6. 3′20″ may you pose. The reciprocal of 3′20″ undone, 18 it gives.
7. 18 to 1 lifted, 18 the upper length. 18
8. to 3 lifted, 54 the lower length {lower length}.
9. ½ of that 36 broken off, 18 (the text writes 17) to 1`12° (=18+54) lifted,
10. 21`36° from 36` (=13`3°+22`57°) undone*, 14`24°.
11. The reciprocal of 1`12° the length undone, 50″ to 14`24° lifted,
12. 12 it gives, 12 to 36 appended: 48.
13. 48 the upper width. 12 to 27 appended,
14. 39 the bar. 12 the lower width it gives.

MKT interpretes the demonstrative ne in the phrase "½ 36 gaz ne 17" as belonging to 17 (18), "½ of 36 broken off, this is 18". Insertion of a deictic particle in such a place is, however, both unmotivated and without predecent (and grammatically probably less correct than the rest of the text). Since, however, a particular stress can be given to a particle from the nominal part of a sentence by placing it after the verb (gaz) (cf. Falkenstein 1959:52 § 36e), and since there are good reasons to give such particular stress when pointing to the 36 left behind already in obv. 5 (36 which should be distinguished from the 36`, the total area, which turn up in the next line), I would rather let the ne belong with 36. Cf. the discussion of the related construction "ba-a-šu ša X" on p. 40 and in note 12.

The translation calls for a few explanatory remarks. I have tried to render Sumerian grammatical forms as precisely as possible**, although the use of Akkadian phrase structures (ša ...) made the enterprise dubious at certain places, and although I did not feel sure whether "infinite" verbal forms should be rendered as such (as I did outside relative clauses), or they were to be read as stenographic logograms for "finite" forms as in other mathematical texts.

As concerns the "semantic" aspect of the

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interpretations of intermediate results, as amount of grain produced, surface, etc.

Further evidence for a "subtractive" interpretation of paṭārum even when reciprocals are dealt with is found in BM 85210,Rev.I,3, 9-10. Here, igi y gál x is used to designate x/y; the process by which it is found is designated by the term zi, "to tear off", while the text uses du_8 when a reciprocal is found. The same or similar constructions are found in BM 85194, Obv.I,45, III,2-3 and Rev. I,28. The first of these places shows that the interpretation of the expression must be "tear off the y'th part of x". (Both texts are in MKT I).

* The sign used is du_8≈paṭārum, in non-mathematical texts "(ab)lösen, auslösen", in mathematical texts used for the finding of reciprocals (as in this text du_8 is used in all other occurrences). The double use of the term in a mathematical text (as also in Str. 362) suggests that the non-technical sense which suits the subtractive process "to detach 21`36° from 36` " so well is also the sense of the term when reciprocals are found: To find a reciprocal is "to take out the n'th part (from 1)". Hence the use of the translation "to undo" in both cases. Cf. addendum p. 75.

** To the best of my very sparse knowledge of the language, and as I cuuld find assistance by Falkenstein (1959) and Deimel (1939). Still, I feel rather confident at the result because a post factum check showed it to be on the whole coincident with Neugebauer's grammatical interpretation.

translation, I have used my standard translations of Akkadian terms combined with the current logographic correspondences of the mathematical texts: gar ≈ šakānum ~ "to pose", gar-gar ≈ kamārum ~ "to take together", etc. For "raising" multiplications I have translated "lift" because the Sumerian term used is not the normal íl ≈ našûm but the conceptually close nim*.

After these remarks on the translation, we can approach the solution of the problem as exposed by the text. I shall do this section by section.

Obv. 1-6. Here the problem is stated. On the diagram can be noticed that the length AC seems to be intentionally perpendicular to the widths, while FD is not. So, there is no doubt on the figure that the length AC is the length spoken of in the following. As it will be remembered, this pointing-out of "good" angles was also characteristic of the field plans, which left even less doubt concerning the distinction between "good" and "bad" corners of the figures.

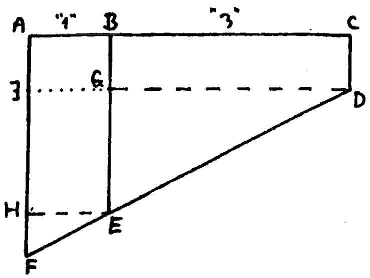

Figure A

In the field plans, dimensions were written on the relevant lines. In our text, the proportionate numbers "1" and "3" are written on the lengths AB-BC. So, the identity of this as the

* The translation is not without problems, but neither is the use of the term itself in the texts. According to current handbooks (ŠL, MEA, ABZ), nim corresponds to intransitive Akkadian verbs: elûm, "(be) high", šaqûm, "be/become high". In our text, however, the use is indubitably transitive, as is my translation.
Of course, the explanation may be that the author of our text (and those of other Strassburg-tablets which use the same term) had a limited knowledge of Sumerian, and translated his Akkadian thoughts wrongly. This would correspond to the use of a quasi-plural-ending in obv. 6, sag-meš, "the widths", where the suffix meš should, according to Falkenstein (1959:§ 18d) belong exclusively with nouns of the animated class, among which "widths" can hardly be counted.

Thureau-Dangin (TMB p. 239) states that "Assez vraisemblablement, nim *= elû III.1" [III.1=Š-stem,"faire monter"]. This would seem to be supported by TMS XIX, 1-2, which has a syllabic i-la-am for multiplication. However, YBC 4608 (MCT p. 49f) alternates between* nim *and syllabic našûm, suggesting that* nim *is simply used as another Sumerogram for this Akkadian word (YBC 4608 and Str. 367 are very closely related). The Susa-text would then simply demonstrate that sumerograms could be spoken directly, without translation into Akkadian (as already suggested by Neugebauer, 1932, in his claim that Akkadianization of Sumerian words and ideograms should be avoided).*

real length is confirmed, as is, more generally, the legitimity of the normal drawing of Babylonian trapezoids and triangles as right-angled figures.

In the verbal statement of the problem will be noticed that the length JF (=36) is not spoken of directly, only as the sum of HF and GE (terms of Figure A): Already at this point, the trapezoid is thought of as consisting of two independent figures - seemingly an adumbration of the method later used to solve the problem. So, the problem seems, as so many Babylonian mathematical problems, to have been constructed primarily as an occasion to use a specific method for its solution.

Obv. 7-12. HF and GE are found by an argument involving the proportionate numbers 1 and 3. The text contains no certain clues to the pattern of thought behind the calculation. We can only presume, from the parallel multiplications by 1 and 3 in rev. 10, that either a real argument of proportionality or an argument of the false-position-type was used - the latter possibility being the one proposed by Vajman (1961:116). The upper length can hardly have been thought of directly as the fourth of the total (we should remember that the normal Babylonian way to think of fractions would be as the fourth, (igi 4 gál-bi) as the fundamental concept, not one fourth as we are led to do by our language and our notation.

Whatever the precise pattern of thought, HF is found to be 9, and GE to be 27.

Obv. 13 - rev. 2. The two partial areas are multiplied by the reciprocals of the proportionate numbers - i.e., both are reduced to length "1" in terms of the scale of these numbers. Afterwards the difference between the two reduced areas is found.

The normal scaling procedures suggests

Another interpretation of obv.13 - rev.2 would be that the average widths of the two partial fields AFEB and BED'C' are found under the supposition that 1 and 3 are the real lengths of these. A similar pattern seems indeed to be followed in YBC 4608 Nº 1 (MCT p. 49, cf. also Vogel 1960:94f). There, however, the average widths are at once doubled, so as to become sums which are then easily manipulated in the head (as stated). Here, no such doubling is made; instead, the inverse area calculation for a trapezoid in rev. 2-4 indicates that a trapezoid, a plane figure, not just a difference between averages is dealt with (difference between averages would not call for the differentiation between "breaking off one half" and "lifting to 30'". ⟸

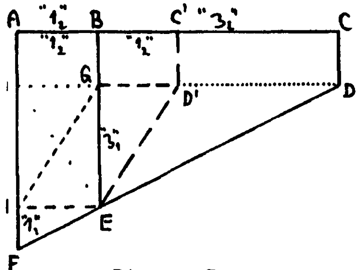

Figure B

that we think of the reductions and the difference concretely, as existing figures. Figure B hows how this can be done.

The left partial field is of course unchanged. The rigth has (in order to become comparable with the left field) its length divided by 3, whereby it is changed into BC'D'E. Since this is equal to ABGH, and also to ABEJ, the amount by which the left field goes beyond the scaled right field is equal to the trapezoid HGEF, as well as to the triangle FEJ,- in both cases to a figure whose surface would be found by the Babylonians by taking 1_1 and 3_1 together, breaking off its half, and raising to 1_2 - where the subscribed "1" and "2" indicate that proportionate numbers for width and length are referrred to - cf. the figure (the reason for this awkward notation will be seen in the following).

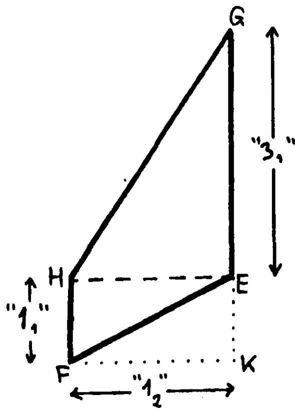

Figure C

Rev. 2-4. Here, a quantity called the "false NU" (or, in Thureau-Dangins reading, "false calculation", cf. p. 105.35 note (**)) is found. The procedure is the inverse of the calculation of the area of the "difference figure" found in the previous section, with 1_1 and 3_1 treated simply as 1 and 3. It appears that the "false NU" is the value which HE ($=1_2$) should possess if the area of the difference figure is maintained as 2`42°, and its widths are really 1 and 3 (i.e., if $1_1=1$).

An alternative interpretation would be that the "false NU" is the area of the rectangle EKFH (Figure C). In the next section, however, evidence against this interpretation will turn up.

The sense of the term NU is unclear. The equivalence with Akkadian ṣalmum, "statue", "figure", "representation", is a vague possibility. Another equally vague possibility might be the approximate homophony between elum/ilum,

"god", which ŠL lists among its logographic values (N° 75,4), and elum, "the upper" or elûm, "(be) high", "ascend" - the latter group of meanings might be connected to the "ascension" of the height HE (cf. muttarittum, see pp. 105.9 and 105.15, the main sense of which refers to the opposite movement)*.

Rev. 4-8 calculates the upper and lower lengths, AB (="1_2") = 18, BC (="3_2") = 54. This is done in what seems a strange way, at least if we believe that the "false NU" is the area of the rectangle EKFH. Indeed, it is known that HF is 9, and so we should only divide 2`42° by 9 (i.e. multiply by 1/9) in order to find HE and thus AB. Instead, it seems, $HE^{-1}$ is calculated as 9/2`42°, and from there HE itself.

This interpretation can hardly be taken earnest as it stands. We must presume that the author of the text had something else than a detour in mind when making his text**. A possible explanation can be found; the awkward subscript numbers were introduced with that explanation in mind.

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* It is relatively unimportant whether the value "god" listed in ŠL is correct Sumerian or a later Akkadian error, if only the error was possible when our tablet was made - we have already seen indications that its Sumerian was a bit homemade. It is even not decisive whether the association should go from "god" to "ascend"/"come forward" (from ilum to elûm) or the other way. If Deimel is right in his claim that the original sense of the sign has to do with procreation, the association to elûm in the sense of "coming forward" is at least as plausible as an association between ṣalmum, "statue e.g. of a god" and ilum, "god", another possible explanation of the values of the sign.

** The detour is even longer than here explained. There is, if we think in terms of the true dimensions of the figure, no reason to calculate our "false NU". Instead, we might have added the real widths 9 and 27 of our difference-figure, have broken into two, and divided the excess area by the resulting 18: This would have given HE directly.

We observe that the area of our excess surface shall be kept constant (as 5`24°), and that a set of mutually reciprocal numbers $9/2`42° = 3'20'' = \varphi$ and $(3'20'')^{-1} = \varphi^{-1} = 18$ turn up. Furthermore, φ^{-1} occurs as a factor multiplying 1_2, i.e. transforming the proportionate number into a real length. All this suggests that we are confronted with a pair of scalings in one dimension, one (transforming the width) with the factor φ, and one (transforming the length) with the factor φ^{-1}.

Precisely such a pair of reciprocal scalings appear to occur in YBC 4608 No 1, cf. Vogel 1960:94f.

Things are, however, a bit more complicated than this. As a matter of fact, φ^{-1} multiplies 1, not the value 2`42° found for the false NU (i.e. for "$1_2$"), and a scaling by factor $\varphi = 9/2`42°$ transforms 2`42°, not 1, into the 9 required. By some argument (which need not be far-fetched, but which is in any case not stated) it is seen that you obtain the same area by putting $1_1$ = 2`42° and 1_2 = 1 as by putting 1_1 = 1 and 1_2 = 2`42°*. Indeed, if one follows what might seem to be the Babylonian habit to interprete "proportionate numbers" as "false values", a "combined false position" like this one requires that one can manage this mutual dependency (which can, in principle, already be seen as a combination of two scalings with reciprocal factors, albeit of a simpler sort).

So, the scaling appears to be made on a figure with 1_1 = 2`42°, which shall be transformed into 9, and 1_2 = 1. If this is the problem, the steps occurring in rev. 4-8 are fully rational and justified. So at least

* It is also conceivable that the "false NU" be from the beginning identical with "1_1", i.e. that it is calculated as the value of HF which would follow if "1_2" were really 1, when the excess area is 5`24°.

a possible explanation of these steps can be found which agrees with normal methods and conceptual habits of the Babylonians, and which, on the whole, follows the text step by step in the problematic passage obv. 13 - rev. 8*.

Rev. 9-10 is more easily explained. Now, JF (=36) and, implicitly, JD (=18+54) are known. From this, the surface JFD is calculated. It is then "undone" from the total surface, and the surface ACDJ is left (cf. Figure **A**).

Rev. 11-15 calculates the width AJ=BG=CD of this rectangular surface by division by the length (1`12°), and finally (by appending JF, GE and nothing) the upper width AF, the bar BE and the lower width CD**.

Fundamentally, the whole problem is "of the first degree". Its treatment, however, is not rhetorical like the "length-width-equations" (see pp. 80ff, 86ff and 92ff). It is treated like other field-partition-problems by the methods known from the geometrical heuristic of the algebra: Scalings, partitions, subtractions and appensions of lines. As previous texts, it confirms the kinship between the geometrical methods of the algebra and the methods of "genuine" geometry as practiced by the Babylonians.

* A related analysis was offered by Vajman (1961:116f). The use of an "excess figure" was also proposed by Vogel (1959:74).

** The explanation of the two last sections coincides with the one already given in MKT.

IM 52301: Algebra, arithmetic, and geometry

While many of the geometrical texts previously dealt with show us algebra and geometry as cognate, the Tell Harmal tablet IM 52 301 shows us algebra, geometrical heuristic and geometry in a more intricate relationship, which may forebode the conceptual changes which were to take place between the Old Babylonian and the Seleucid age.

Like most mathematical tablets from Tell Harmal, the text is dated to the latest part of the Old Babylonian period. The part which concerns us here runs as follows*:

Obv. 1. When 1`40° the upper length, its counterpart (meheršu) being lost, the upper width
2. over the lower width 20 goes beyond,- how much my lengths**?
3. You, by your making, 1°30´ pose, break, {cross:}
4. 45´you see. The reciprocal of 45´ undo, 1°20´ you see. 1°20´ to 40`, the surface, raise:
5. 53`20° you see. 53`20° double: 1``46`40° you see. 1``46`40°, that your head
6. retain. Turn back. 1`40° the upper length and 20 which the upper width
7. over the lower width goes beyond, take together: 2` you see.
8. 2`break: cross: 1`` you see. 1`` to 1``46`40° append:
9. 2``46`40° you see. The side of 2``46`40° search, 1`40° you see.
10. To 1`40° your side, 1` which you crossed to 1`40° append:
11. 2`40° you see. From 2`40° which you saw, 1`40° the upper length cut off,
12. 1` is left. The lost length, 1`. Break, 30 you see, 30 the counterpart (mehrum)

* Originally, the text was published by Baqir (1950a). I follow the improved transliteration due to Gundlach & von Soden (1963:252f).

** Here as elsewhere (but not everywhere) in the text, uš, "length", is used in the generalized sense of "side".

13. lay down. 20 which width over width goes beyond, break:
14. 10 you see. 10 to one 30 append: 40 you see; from the second 30
15. cut off, 20 you see, 20 the lower width. Such the being-made.

---

16. When to two-thirds of the accumulation of upper [2]
17. and lower width[1], 10, to my hand (i.e. at my disposition), I have appended, (and so) 20 the length I have built; (when) the upper[2]
18. width[1] {over} over the lower 5 goes beyond;
19. (and when) the surface is 2`30°; how much (then) are my lengths? You, by your making, 5 which goes beyond
20. 10 which you appended, 40´ the two-third, the aramaniātum*, inscribe:
21. The reciprocal of 40´, the two-third, undo, 1°30´ you see, 1°30´ {break:
22. 45´ you see, 45´**} to 2`30° the surface raise: 3` 45° you see.
23. 3`45° double, 7` 30° you see, 7`30° that your head
24. retain. Turn back: The reciprocal of 40´, the two-third, undo,

Rev. 1. 1°30´ you see. 1°30´ break, 45´ you see, to 10 that you appended
2. raise, 7°30´ you see. {7°30´ that your head retain.
3. Turn back. The reciprocal of 40 undo: 1°30´ you see, 1°30´ (text: 1°40´) break:
4. 45´ you see, to 10 that you appended raise, 7°30´ you see}.

NB: The statement that the length is 20 is not intended to be used for the solution - the whole problem will be of the first degree if it is used. Instead, the "20" seem to be a reminiscence of the habit to refer to the quantities involved in a problem by the number which measures them, i.e., "20" is used to identify the length (cf. TMS XVI, pp. 86ff above). In rev. 17, the value of the length is calculated regularly from the other relations stated in the beginning of the problem.

* Mathematically, the aramaniātum is the number which multiplies the sum of the widths. Grammatically, it is a plural. and etymologically it appears to descend from a-rá. So, the term appears to suggest (two) identical factors multiplying the members of a sum. In agreement with this, von Soden (1952a:50) proposes tentatively the word to be a loan-word from Sumerian ara-man, "times"-"two", i.e. "factor of both".

** At this point, Gundlach & von Soden insert a passage in <>, in an attempt to make mathematical sense of the passage. Since, however, the whole passage in {} has slipped in by error from rev. 1, no restitution is required since no mathematical sense is to be expected.

5. 7°30′ the counter{which}-part lay down, cross:
6. 56°15′ you see. 56°15′ to 7`30° which your head
7. retained append, 8`26°15′ you see. The side
8. of 8`26°15′ search, 22°30′ its side. From 22°30′
9. the side, 7°30′ your crossed (takīltum) cut off,
10. 15 is left. 15 break, 7°30′ you see, 7°30′ the counterpart lay down.
11. 5 which width over width goes beyond, break:
12. 2°30′ you see. 2°30′ to one 7°30′ append:
13. 10 you see; from the second 7°30′ cut off,
14. 10 is the upper width, 5 is the lower width.
15. Turn back: 10 and 5 take together, 15 you see.
16. The two-third of 15 take: 10 you see, and 10 append:
17. 20 your upper length. 15 break: 7°30′ you see.
18. 7°30′ to 20 raise, 2`30°, the surface, you see.
19. Such the being-made.

20-24. *Fragmentary list of constants;left out as uninteresting in the present context.*

Edge 1. When a surface of unequal lengths (i.e. sides - uš la mitharūti, a plural genitive). You, the reciprocal of 4 undo:

2. The totality (napharum - book-keeping term) of the lengths, that they say it to you; to the totality of the lengths raise:
3. To the 4(?) directions of the wind inscribe. As much as went out, cross: From inside,
4. the surface you tear off.

A number of insertions and omissions - the former indicated by {___} - characterize the tablet. They must presumably be due to careless copying or dictation:

- Obv. 3, "šu-ta-ki-il" has crept in from another place where halving is followed by crossing (to wit obv. 8).

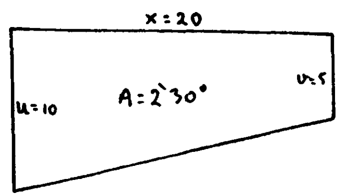
- Obv. 21-22, a passage anticipates rev. 1, inspired by the common phrase "The reciprocal of 40′ ... 1°30′".

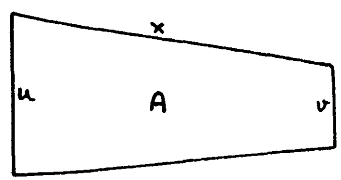
- Rev. 2-4, a long passage repeats sign for sign (apart omission of a sign 10 in 1 40, the omission of the word ši-ni-pé-tim and the introduction of a -ma) a passage from obv. 23-rev. 2, because of the repeated "7 30 you see".

- Obv. 2 (or nearby), a statement of a relation between sides is omitted - cf. below.

- Obv. 12, a passage calculating in some way the sum of the widths is omitted.

- Finally, the inscription of the edge seems to be somehow out of order - cf. below.





These numerous errors take away part of the credit which can be given to any conclusion drawn from the tablet. However, as most of those errors which are definitely established consist in repetition or omission of whole passages, we may assume that the words which are found in the tablet belong legitimately to the text.

Problem I (obv. 1-15) is insufficiently stated; this appears from a mathematical analysis of the problem, but also from obv. 3 where a number 1°30′ pops up from nowhere. As demonstrated by Gundlach & von Soden, problem II (obv. 16-rev. 19) provides the clue to what is going on. So, I shall begin the investigation at obv. 16.

The text deals with something possessing an upper and a lower width (obv. 16f) and one length (obv. 17), which from its use in the area calculation is regarded as a "true" lenght, and which in rev. 17 is spoken of as the upper lenght. So, a quadrangle is dealt with, presumably an approximately right trapezoid (upper diagram); alternatively, the trapezoid is equilateral, and the "upper" of rev. 17 is due to a slip of mind (lower diagram).

Given are the following relations:

$$\frac{2}{3}(u+v) + 10 = x \ ; \quad u-v = 5 \ ; \quad A\ [= x\cdot\frac{u+v}{2}] = 2`30°$$

The procedure appears to follow this scheme ($z=u+v$):

$$x\cdot\frac{z}{2} = 2`30^\circ \qquad x = \frac{2}{3}z+10$$

$$(\frac{2}{3}z+10)\cdot\frac{z}{2} = 2`30^\circ$$

$$(z+\underline{\frac{3}{2}\cdot 10})\cdot\frac{z}{2} = \frac{3}{2}\cdot 2`30^\circ = 3`45^\circ \quad (\text{---: Not yet performed})$$

$$(z+\frac{3}{2}\cdot 10)\cdot z = 2\cdot 3`45^\circ = 7`30^\circ$$

(*) $$z^2 + (\tfrac{1}{2}\cdot\frac{3}{2}\cdot 10)\cdot 2z = 7`30^\circ \quad (\text{solved as "square+sides"})$$

$$z^2 + 7^\circ30'\cdot 2z + (7^\circ30')^2 = 7`30^\circ+(7^\circ30')^2 = 8`26^\circ15'$$

$$z + 7^\circ30' = \sqrt{8`26^\circ15'} = 22^\circ30', \text{ whence}$$

$$z = 22^\circ30'-7^\circ30' = 15.$$

So, $u+v$ is known, and u and v are found separately from $\frac{u+v}{2}$ and $\frac{u-v}{2}$. Finally, x and A are calculated, the latter as a control (the former only halfway).

One can ask to what extent the above chain of symbolic transformations maps the precise thought of our Babylonian author, especially whether he was really working in terms of $u+v=z$ as the unknown quantity, or he would have formulated himself when asked in terms of $\frac{u+v}{2} = \frac{z}{2}$, i.e. in terms of the average width, as would seem more normal to Babylonian thought.

The answer appears to be that the basic entity was definitely z, not $\frac{z}{2}$. Had the latter been the case, $3`45^\circ$ would not have been doubled but halved; in that way, we would found the value $\frac{z}{2} = 7^\circ30'$ directly, not via $2\cdot\frac{z}{2} = 15$. Instead, the calculation passes a point (*) where, according to other well-known examples of Babylonian mathematical thought, $7`30^\circ$ must have been thought of as a "square plus sides" with z^2 as the square and z as the corresponding side.

One could also ask whether the problem was necessarily seen as a problem in one variable (z), or it was perhaps seen as a

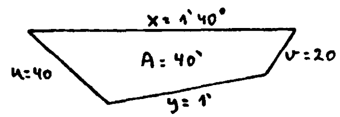
length-width-problem: If we put $z+\frac{3}{2}\cdot 10 = P$, $z = Q$, we have $P-Q = \frac{3}{2}\cdot 10$, while (*) becomes $P\cdot Q = 7`30^\circ$.

Mathematically, this is of course equivalent, and the numbers operated upon would be the same. However, had our author supported his string of calculations by this conceptualization, we would expect him to calculate explicitly $P-Q = 15$, and them to break it into two (cf. TMS XIII, pp. 7 and 84 above, and YBC 6967, p. 11). Instead, $\frac{3}{2}$ is broken, and the result is raised to 10 (rev. 1-2). So, although length-width-problems are more common than square-plus-sides-problems, we can be almost sure to be presented here with argumentation of the latter type.

Cf. a related discussion above, p. 59, whether one of the siege-ramp problems was solved in one or two variables.

Before commenting further upon the procedure and the formulations of problem II, I shall try to trace the meaning of problem I on the basis provided by the above interpretation of the former.

In problem I, an upper and a lower width occur, as well as an upper and a lower length; together, they determine an area. So, we are dealing with a quadrangle of unequal sides (as it turns up from the results) as shown in the figure to the left.



Once more, we may notice that the quadrangle is, when drawn in correct proportions, seen to be too skew for the area formula employed to give a reliable result (and of course, that the real figure is undetermined).

As mentioned above, the statement of the problem is incomplete. Because of the $1^\circ30'$ popping up in obv. 3 we may assume that the lacking statement is somehow analogous to the first equation given for problem II, and we may try to work backwards from obv. 12, where y is found to be $1`$. The analogy with problem II leaves little real choice for the interpretation of the steps:

$$y = 1`$$

$$x+y = 1`+1`40^\circ = 2`40^\circ \text{ ; putting } z = x+y,$$

$$z-1` = 2`40^\circ-1` = 1`40^\circ$$

$$(z-1^{`})^2 = (1^{`}40^{\circ})^2 = 2^{``}46^{`}40^{\circ}$$

$$z^2-2^{`}\cdot z+1^{``} = 2^{``}46^{`}40^{\circ} \text{ ; putting } a = 2^{`} = x+(u-v)$$

$$z^2-a\cdot z = 1^{``}46^{`}40^{\circ}$$

$$\tfrac{z}{2}\cdot(z-a) = 53^{`}20^{\circ}$$

$$(*) \quad \tfrac{z}{2}\cdot(\tfrac{1}{2}\cdot\tfrac{3}{2})(z-a) = 40^{`} = A$$

Since $A = \frac{x+y}{2}\cdot\frac{u+v}{2} = \frac{z}{2}\cdot\frac{u+v}{2}$, (*) implies that the missing information must have been equivalent to

$$u+v = \tfrac{3}{2}\cdot\{(x+y) - [x+(u-v)]\} ,$$

i.e. to something like "I have *taken my upper length and as much as my upper width goes beyond my lower width together*: as much as the *accumulation of the lengths* goes beyond I have *raised to 1°30′*: That is as much as the accumulation of the widths". The part of the expression which is underlined is, together with the parenthetical structure of the symbolic expression, determined with great probability from the precise progress and formulation of the calculation. On the other hand, the mathematically equivalent expression found by Gundlach & von Soden does not (and is neither claimed nor intended to) fit the text in this manner.

So, problem 1 starts from an (unformulated) equation of the type

$$\tfrac{z}{2}\cdot[\alpha(z-a)] = A ,$$

while problem II starts from

$$\tfrac{z}{2}\cdot(\alpha z+a) = A .$$

In neither case, the reduction follows the characteristic procedure which we met earlier (in BM 13901 N° 9, pp. 12f above), which would lead to an equation

$$(\tfrac{\alpha}{2}z)^2+2\beta(\tfrac{1}{2}\tfrac{\alpha}{2}z) = \Gamma .$$

Instead, the reduction sought for is to the form

$$z^2+\gamma z = \Delta ,$$

i.e. the form current in Medieval (Arabic and

Latin) rhetorical algebra. In problem I, this may not be remarkable. The specific form of its initial equation nullify the reasons which make the "Babylonian" method easier to handle than the "Arabic" method in a geometrical representation. But the equation of problem II is precisely of the character discussed above on pp. 14f.

It will also be remarked that the products in obv. 5 and obv. 23 are to be retained by the head, as otherwise required in transformations of the first degree (cf. p. 82 above, and *passim*).

The two observations taken together sugggest that the transformations of the basic equation into a normalized equation of type "square and sides" must have been made by mental, rhetorical argument, with no support in geometry (imagined or drawn). So, since the text is very late Old Babylonian, the trend toward arithmetization of thought postulated above (p. 62) on bad evidence and thus retracted in the marginal note, seems to receive new, more reliable support.

Maybe the use of u+v instead of $\frac{u+v}{2}$ is also pointing toward a gradual emergence of the patterns of thought found fully developed in the Seleucid texts. In any case, the operation on x+y and x-y instead of $\frac{x+y}{2}$ and $\frac{x-y}{2}$ was discussed as a deviation from older habits in connection with BM 34568 N° 9 above (p. 101).

Even though there are perhaps indications in our text of a certain development toward arithmetization, other features point toward continuation of earlier ways. The conceptual distinction between "raising" and "crossing" is upheld in the solutions of the normalized equations. One can compare with al-Khwârizmî's *Algebra*, where precisely the solutions to the basic normalized second-degree equations are proved correct by geometrical heuristic arguments.

Certain features of the terminology may also convey the message that the geometrical background is fading away but not yet forgotten. The term <u>banûm</u>, "to build", used in other mathematical texts when a surface is constructed by "crossing", turns up in obv. 17 at a point where only first-degree operations* (and thus no geometrical construction) have been performed. <u>Meḫrum</u>, "counterpart" (translated elsewhere, e.g. p. 11 above, as "equal"), is used in a clearly geometrical sense in obv. 1. In obv. 12 and rev. 10 it turns up in its well-known normal function in the solution of a second-degree equation. But it is also found in rev. 5, in a position where the term is not currently expected. The reason is that a crossing is to be performed on only one magnitude - 7°30′ is found not by breaking 15 (which would yield two "copies") but by raising 45′ to 10, the 45′ being found by breaking 1°30′. In other words: "Breaking" is still the term used, but what goes on is arithmetical halving, not a geometrically conceptualized bisection.

The generalized use of uš, "length", in the sense of "side", is probably not an indication of waning geometrical conceptualization. Still, it appears to indicate a slackening of the connexion between field surveying and quadrangle geometry.

The <u>aramaniātum</u>, "factor of both", a term derived from a-rá and applied to a factor in a "raising", may, finally, perhaps be taken

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* This deviation from normal terminology was pointed out by von Soden in his first commentary to the text (1952a:50).

as a symptom of an emerging conflation of these two multiplications - a merging which is also fully consummated in the Seleucid texts.

A final term to be discussed is <u>takīltum</u> - not because of its importance for this text but because the present text has (in error, I believe, and as I intend to demonstrate) been taken as a support to the interpretation of that term as "that which is held" (Baqir 1950a:148). Cf. the addendum to note 5a, p. 126 below.

The <u>takīltum</u> occurs as 7°30′ in rev. 9. Mathematically, it refers to the 7°30′ "seen" in rev. 2 and crossed in rev. 5. As shown above, this 7°30′ is <u>not</u> retained by the head - the words which seem to indicate this are merely a repetition of earlier phrases referring to the number 7`30° calculated in obv. 23, the function of which is quite different - it appears again in rev. 6, where it is spoken of precisely as that "which your head retained". The 7°30′ spoken of as takīltum, on the other hand, was crossed; so, nothing would be more natural than to speak of it as "that which was crossed".

On the other hand, it would be very unpractical to refer to both 7`30° and to 7°30′ as that "which was retained". Even without this confusion, the difference in the order of magnitude must have been puzzling (and, as we see in rev. 5-7, our author managed to keep clear of the confusion).

Parallel passages, finally, support the connexion to "crossing". In the same tablet, obv. 10, we have a very close parallel; yet, instead of <u>takīltum</u> the text speaks of "that which you crossed", <u>ša tuštakilu</u>. In YBC 6967 (see p. 11 above), the term is also found (rev. 1) referring to a number 3°30′ which is not retained according to the words of the text (and which, in this earlier text with its clear respect for the conceptual distinctions of the geometrical heuristic, could not be expected to be retained in the head), but which has, once again, been "crossed" at its previous appearance, and which, furthermore, occupies precisely the same position inside the solution of the equation.

So, all evidence in our text points in the direction that takīltum should not, and maybe could not, be understood as "the thing retained".

The edge

The text on the edge was interpreted by Bruins (1953:242f, 252) as a description of "Heron's approximation" to the square-root of a non-square number, in term very much reminding of the geometrical heuristic.

Bruins' interpretation is, however, based "auf eine Übersetzung, die nicht dem Wortlaut des Textes entspricht" - to quote Gundlach & von Soden (1963:259). So, in order to clear away false support for my interpretations of Babylonian algebra, I shall expound Gundlach & von Soden's interpretation, which I find convincing, together with some supplementary reasons to uphold it.

In the two problems it was made clear that "lengths" is used at several places in the generalized sense of "sides". So, thanks to the syllabic mitharuti, we are sure to deal with "a surface of unequal sides" - apparently a quadrangle of unequal sides. The quantity which in raised in l. 2 to the totality (i.e. sum) of the sides must be the reciprocal of 4 asked for in l. 1. The "inscription to the four winds" (i.e. "in the four directions") comes in in l. 3 as a red herring (and as the fundamental basis for Bruins' interpretation) between the

raising and the reference to its result. Probably it has to do with the first part of l. 2: It is required that the lengths of the four sides be inscribed on the relevant places in a diagram.

Symbolically, the following expression appears to be calculated:

$[\frac{1}{4}(x+y+u+v)]^2-A$, where by assumption $A=\frac{x+y}{2}\cdot\frac{u+v}{2}$,

which reduces (but is not reduced in the text) to

$[\frac{1}{2}(\frac{x+y}{2}-\frac{u+v}{2})]^2$.

It may seem puzzling that the text is so fragmentary: Why should it prescribe such an isolated piece of computation? On the other hand, the tablet contains another fragment, the list of constants in rev. 20-24. So, the fragmentary character of the four lines need not puzzle us too much.

One might of course try whether the text could be meant to describe the area formula for an irregular quadrangle. One could observe that the ordinary book-keeping term napharum may designate a subtotality, while the summa summarum is naphar naphar (AHw II,737a), and one could try whether l. 2 could imply that the sum of two sides (x+y) is raised to the sum of the other sides (u+v). If l. 1 is to have any meaning, the resulting ¼ should then also be raised to the product, which would give us the area of the quadrangle. But ¼ is not mentioned in connexion with any multiplication. Furthermore, the "crossing" in l. 3 followed by a "tearing-off" of the surface asks for the calculation of A^2-A, which is not only without purpose but also without meaning for dimensional reasons. To save this hypothesis, the "crossing" in l. 3 should be interpreted as a repeted statement of the "raising" in l. 2 (or the forgotten multiplication by ¼), and the tearing-off in l. 4 as a misunderstanding of a relation of equality. Gross errors of <u>this</u> kind are, however, not characteristic of the rest of the tablet. So, the text on the edge can hardly be a description of the area formula for the irregular quadrangle (and if it could, it would be a formula different from that used in problems I-II, which is $\frac{x+y}{2}\cdot\frac{u+v}{2}$, not $\frac{1}{4}(x+y)\cdot(u+v)$).

With this somewhat enigmatic fragment of an attempt at describing a procedure in abstract terms, halfway between the geometry of irregular quadrangles and the algebra of binomials and polynomials I shall end the appendix on the Old Babylonian "real geometry".

ABBREVIATIONS

ABZ	Rykle Borger 1978, <u>Assyrisch-Babylonische Zeichenliste</u>.
AHw	Wolfram von Soden 1965, <u>Akkadisches Handwörterbuch</u>.
AkSyll	Wolfram von Soden 1948, <u>Das akkadische Syllabar</u>.
BAG	Carl Bezold 1926, <u>Babylonisch-assyrisches Glossar</u>.
CAD	<u>The Assyrian Dictionary of the Oriental Institute of the University of Chicago</u>.
GAkGr	Wolfram von Soden 1952, <u>Grundriss der akkadischen Grammatik</u>.
HAHw	Wilhelm Gesenius 1915, <u>Hebräisches und Aramäisches Handwörterbuch</u>.
MCT	Otto Neugebauer & A. J. Sachs 1945, <u>Mathematical Cuneiform Texts</u>.
MEA	René Labat 1963, <u>Manuel d'épigraphie akkadienne</u>
MKT	Otto Neugebauer, Mathematische Keilschrift-Texte. I: 1935; II: 1935a; III: 1937.
ŠL	Anton Deimel 1925 (-1937), <u>Šumerisches Lexicon</u> I-III
TMB	F. Thureau-Dangin 1938, <u>Textes mathématiques Babyloniens</u>.
TMS	Evert M. Bruins & M. Rutten 1962, <u>Textes mathématiques de Suse</u>.

<u>Note on assyriological aids</u>

During the preparation of the first version of the text, I used ABZ, AHw, BAG, CAD and GAkGr for lexical questions, together with the glossaries of MCT, MKT II-II, TMB and TMS. (For geographical reasons, I was only able to draw on AHw and CAD for some final checks). For questions pertaining to Akkadian language and grammar, I used GAkGr and Riemschneider 1969,- for script and signs, AkSyll and ABZ.

For addenda, corrections and the whole appendix, I have also had continuous access to ŠL I-III, to MEA and, thanks to the generosity of Roskilde University Library, to AHw. For questions of Sumerian language and grammar, I used Falkenstein 1959 and Deimel 1939 during this phase.

Table 1: The standard translations, alphabetically ordered

For a number of terms with a fully or partially technical function in the texts, a standard translation into English has been fixed in order to allow identification of the terms used in the original texts. As it will be seen, no differentiation is made between Akkadian and Sumerian words etc., when logographic equivalence is securely established.

The table can be used as a key to Table 2, where the original terms are listed together with references to their occurrences in the translated texts and to passages where they are discussed. Entries to this table are marked below by a superscript[S].

"accumulation", kimrātum[S], from kamārum[S].

"again", atur[S], from târum[S]

"append", waṣābum[S]~daḫ

"arrive", sanāqum[S]

"as", inūma[S]

"ask", šâlum[S]

"bar", pirkum[S]

"being-made", nēpešum[S], from epēšum[S]

"break away (from a measuring stick)", ḫaṣābum[S]

"break off ½"/"break into two"/"break into two", ... ḫipûm[S]

"bring (to a place)", wabālum[S]

"build", banûm[S]

"bundling", makṣarum[S], from kaṣārum[S]

"by means of", ina[S]

"come up (as a result)", elûm[S]

"correspond to", maṣûm[S]

"count, the", manātum[S]

"counterpart", meḫrum[S]

"cross" šutākulum[S]

"cut away", kašātum[S]

"cut off", ḫarāṣum[S]

"descendant", muttarittum[S], from warādum[S]

"diagonal", ṣiliptum[S]

"double"/"'double'", eṣēpum[S]

"equal, the", meḫrum[S]~DUḪ[S]

"extension", waṣūbum[S]~KI.GUB.GUB[S], from waṣābum[S]

"draw", nadûm[S]

"false", lul[S]

"field part", tawirtum[S] (~ íd?)

"find out", paṭārum[S]

"first ... second", "ištēn ... šanûm"[S]~ 1(kam) ...2(kam)

"from", ina[S]

"give", nadānum[S]~sum

"give reciprocally", šutākulum[S], cf. akālum[S], takīltum[S]

"go", alākum[S]~rá (to distinguish from the specific use of this in X a-rá[S] Y", "X times Y")

"half of that which is X", ba-a-šu ša X[S]

"he said", iqbu (subj.), from qabûm[S]

"head retain, that your", rēška likil[S]

"inscribe", lapātum[S]

"inside", libbi[S], from libbum

"lay down", nadûm[S]

"leave", ezēbum[S]

"length", šiddum~ uš[S]

"lift", nim[S]

"lost, being", ḫaliq[S], from ḫalāqum

"lower", ki[S]

"name", šumum[S]

"next", asaḫḫir[S] (literally "I turn around"), from saḫārum[S]

"no further it go!", la watar[S], from watārum[S]

"one ... the other", ištēn ... ištēn[S]

"over ... go beyond", eli ... watārum[S]~ugu ... dirig, cf. watārum[S]

"place, in the X-th", ki-X[S]

"pose", šakanum[S]~gar (~U.UL[S]?)

"procedure", epēšum[S]

"raise", našûm[S]~íl

"raise against each other (as counterparts)", šutamḫurum[S], cf. maḫārum[S]

"reciprocal of X" igi X[S]/igi X gál-bi

"remain (behind)", aḫārum[S]

"remaining thing, the" šapiltum[S], from šapālum[S]

"retain", kullum[S]

"scatter", sapāḫum[S]

"search", šâlum[S]

"see", amārum[S], cf. "you see"

"side", íb-si$_8$[S], íb-si

"since", aššum[S]

"become small(er)", maṭûm[S]

"so mach as", mala[S]

"span", cf. akālum[S], šutākulum

"square figure", mitḫartum[S], from maḫārum[S]

"surface", eqlum~a-šà[S]

"take", laqûm[S]

"take away", tabālum[S]

"take together", kamārum[S]~gar-gar ~UL.GAR

"tear off", nasāḫum[S]~zi

"the same as", kīma[S]

"the same as (there is) <of> X" kima X<-im>[S]

"thing to which was given", takīltum[S], cf. šutākulum[S]

"times", a-rá[S] (~GAM[S] in Seleucid texts?)

"to", ana[S]

"together with", itti[S]

"totality", napḫarum[S], from paḫārum[S]

"trapezoid", sag-ki-gu$_4$[S] (~abusamikum[S]?)

"triangle", sag-dù[S] ~santakkum

"true", kinum[S] ~gi-na

"turn (into a frame)", NIGIN[S] (~LAGAB[S]?)

"twice, until", adi šinišu[S]

"undo", paṭārum[S]~du$_8$

"until", ana[S]. In the connection "until twice" adi (šinišu)[S]

"upper", an[S]

"what", mīnûm[S] ~ennam

"when", šumma[S]

"width", pūtum ~sag[S]

"you see", tamar[S], from amārum[S] *(should be tammar)*.

":" translates -ma. The remaining punctuation depends on the global interpretation of the texts and has no counterpart of its own in these.

Table 2: Akkadian and Sumerian word list, alphabetically ordered

Below, terms with a more or less technical meaning or technical function in the mathematical texts are listed with translations and references to occurrences in the translated Babylonian texts and to passages where they are discussed. In certain cases, cross-references to related terms, proposals for a better standard-translation to be used in the translation of mathematical texts or a short commentary is given.

The translations are not intended to cover the full range of the terms translated, only to make clear the values relevant for the understanding of mathematical texts. My proposals for standard translations are underlined (as far as possible, I restrict myself to one proposal for each term; in some cases, notably the prepositions, this would, however, result in completely illegible English, and therefore several translations are underlined).

The most important technical terms are indicated typographically(/). References to occurrences in the texts are listed under T, while references to the commentary are indicated by C.

abusamikum (~sag-ki-gu$_4$^{S}?), a (normally?) specific trapezoid. Leave untranslated.
C: 105.14.

adi šinišu, "until twice" (literally "until its second").
T: 33.

aḫārum, "remain behind".
T: 105.25.

akālum, "eat", "cover (a length or a surface)". Cf. šutākulum^S, takīltum^S, ku^S.
C: 11, 105.24.

/ alākum~rá, "go". Functionally close to eṣēpum^S (as a supplement or a complement). To keep apart from the specific X a-rá^S Y, "X times Y", of the multiplication tables (etc.).
T: 92.

amārum, "see". Cf. tamar^S, "you see".

/ an, "upper" (length, width, surface. etc.). Cf. elûm^S.
T: 105.24f, 105.34ff, 105.43ff.
C: 52.

/ ana, "to", "until". The prepositional complement to waṣābum^S, našûm^S, often šutākulum^S, eṣēpum (when 'doubling' beyond 2 is meant), etc.
T: All translated texts.
C: 18, 64.

/ a-rá (~GAM^S), "times". Cf. commentary under alākum^S~rá.
T: 31ff.
C: 31, 37f, 64f, 101.

arakarûm, cf. commentary. Leave untranslated.
T: 105.25f.
C: 105.25, 105.27ff.

asaḫḫir, "I turn around" (the translation pp. 31ff used "next"). Marks a break in the exposition. From saḫārum^S.
T: 31ff, 105.25f (you ...).
C: 31.

/ a-šà~eqlum, "surface".
T: 11, 12f, 16f, 20, 22, 31ff, 49, 80f, 92f, 105.6, 105.15, 105.17, 105.24f, 105.43ff.
C: 19, 35, 50, 68, 97, 105.7ff, 105.29, 105.34, n.15.

/ aššum, "since".
T: 16, 80, 87, 92, 105.16.

atur, "I turn back" (the translation of pp. 31ff used "again"). Marks a break in the exposition. From târum^S.
T: 31ff, 105.45 (imperative).
C: 31.

ba-an-da (factor required to obtain a given product, i.e. "quotient"?). Leave untranslated.
T: 21.
C: 21.

ba-a-šu ša X, "half of that which is X". Cf. ne^S.
T: 31, 33.
C: 40f, 69, 105.36, n.12, n.17.

/ banûm, "build", "manufacture", "produce", "create".
T: 16, 31ff.
C: 83, 105.33, 105.51.

/ daḫ~waṣābum^S.

/ dirig~watārum^S.

/ du$_8$~paṭārum^S.

/ DUḪ: 1) ~meḫrum^S. 2) =du$_8$~paṭārum^S.
1)C:7, n.5, n.35.

/ eli ... watārum~ugu ... dirig, "over ... go beyond", cf. watārum^S.
T: 11, 16f, 20, 31ff, 87, 105.15ff, 105.25, 105.34f, 105,43ff.
C: (8), 64.

elûm, "be/become high", "go up". Cf. nim^S, an^S. With dative suffix "come up (for you as a result)".
T: 105.25f.
C: 105.25.

/ ennam~mīnûm^S.

/ epēšum~kì, "proceed", "make". As a noun "procedure", "making" (the above translations use "procedure"). Used in the expression "your making" before the start of the description. Cf. nēpešum^S.
T: 31ff, 105.35, 105.43f.
C: 31, 105.33.

/ eqlum~a-šà^S

/ eṣēpum~tab, "double", "'double'" (in case of repetitions beyond twice).
T: 16f, 80f, 105.44.
C: 53, 55, 64f.

/ ezēbum, "leave" (by naṣaḫum^S).
T: 16, 80, 105.16f, 105.43, 105.45.

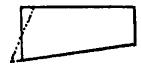
/ GAM (~a-rá in Seleucid texts?).
T: 102.
C: 102.

/ GAR, a measure of length (12 cubits, c. 6 m.). Should be written nindan, cf. Powell 1972:198f.
T: 20.
C: 26, 52, 54.

/ gar~šakānum^S (~U.UL^S?).

/ gar-gar~UL.GAR~kamārum^S?

gíd~arākum, "long"/"prolonged". The right translation cannot be decided from the very few occurrences, which all describe the "long" (as opposed to the short) length of an irregular quadrangle (in spite of the preliminary hypothesis of MKT I, 292). The factitive interpretation would mean that an irregular quadrangle is thought of as obtained from a trapezoid by "prolonging" and shortening". Cf. lugúd^S.
T: 105.25f.
C: 105.29.



However, in IM 55 357, 1, uš-gid designates the hypotenuse of a right triangle, and hence the "long" and not the "prolonged" side (text in Baqir 1950:41).

/ gi-na~kinum^S?

ḫaliq, "being lost" (from ḫalāqum, "perish", "disappear", "flee"). Used to designate a quantity as unknown.
T: 105.43.

/ ḫarāṣum, "cut off".
T: 32, 34, 105.25f, 105.43ff.
C: 31, 64, 67, 105.32.

ḫaṣābum, "break away" (e.g. a fraction from a measuring reed).
C: 64.

/ ḫipûm, "break" (off ½, into 2, etc.).
T: 7, 11, 13, 31, 33f, 49, 80, 105.6, 105.16f, 105.25f, 105.35, 105.43ff.
C: 64, 90, 105.36, 105.38.

/ íd~tawirtum^S?

/ íb-si$_8$, íb-si, "side" (of square), eventually identified with the whole figure. A translation (proposed by Jöran Friberg) which would cover both meanings, and which points to the "equality" inherent in si$_8$, would be "equilateral".
T: 7, 11, 13, 16, 20f, 32ff, 49, 80, 93, 105.16, 105.25, 105.43, 105.45.
C: 8, 49, 64, 67, n.3a, n.9a, n.23.

/ igi X (gál-bi) 1) "the reciprocal of X". 2) "The X'th part". (The conceptual distinction is expressed differently in different texts, but since there is a distinction it should be upheld in translations). Cf. paṭārum[S].
T: 13, 16, 20f, 32, 81, 84, 87, 93, 105.16, 105.25f, 105.35, 105.43ff.
C: 31, (75), 105.34f.

/ igûm/igibûm. Leave untranslated.
T: 11, 80f.
C: 10f, 80.

/ íl~našûm[S]

/ ina, "in", "from", "by", "by means of". The prepositional complement to naṣāḫum, ḫarāṣum and (at times) šutākulum etc.
T: 7, 11, 12f, 16, 32ff, 86f, 92f, 105.6, 105.16f, 105.25f, 105.34, 105.36, 105.43ff.
C: 16, 19, 19a, 64.

inūma, "as".
T: 92.

/ ištēn ... ištēn, "one ... the other". Cf. ištēn ... šanûm[S].
T: 11, 80.
C: n.6.

/ ištēn ... šanûm ~ 1(kam) ... 2(kam), "the first ... the second", cf. ištēn ... ištēn[S]
T: 7, 11, 20f, 32f, 49, (93), 105.44.
C: 9f, 67, 70, 77, 105.26, n.4, cf. n. 6.

/ itti, "together with". A possible prepositional complement to šutākulum.
T: 17, 92f.
C: 19a, 64.

/ kam, (mostly) an ordinal suffix. Cf. išten ... šanum[S].
C: 105.26, n.4.

/ kamārum~gar-gar, "pile", "heap up", "take together". Cf. kimrātum[S].
T: 20, 31ff, 49, 80f, 87, 92f, 105.25f, 105.35, 105.43, 105.45.
C: 24, 64f, 80, 92, 98, n.11.

kaṣārum, "bind/bring/put together". Cf. makṣarum[S].

kašātum, "cut away" (from a quantity bought).
T: 7, 84.
C: 64.

/ ki, "lower" (length, width, surface, etc.). Cf. šapālum[S].
T: 105.24, 105.26, 105.34ff, 105.43ff.
C: 52,

ki X, "in the X'th place".
T: 105.26
C: 105.26.

/ kì~epēšum[S].

KI.GUB.GUB (~waṣūbum[S]?).
T: 92.
C: 97.

/ kīma, "as much as".
T: 16.

/ kīma X-im, "as much as (there is) of X".
T: 87.
C: 27, 90.

/ kimrātum~UL.GAR, "accumulation", from kamārum[S]. Probably a plural, and so, a translation should be looked for which reminds that the "things piled up" and not the "pile" itself are thought of. "Things taken together"?
T: 31ff, (80), 92f.
C: 31.

/ kinum~gi-na, "true".
T: 32, 87, 105.6.
C: 35, 68, 71, 91, 105.7ff.

kú~akālum[S], ~šutākulum[S].
T: 105.24.
C: 11, 105.24.

kullum, "retain, "hold". Cf. reška likil[S].
T: 87.

la watar, "no further it go!" From watārum[S] (prohibitive stative, cf. GAkGr § 81k).
T: 32f.
C: 31, 39.

/ LAGAB, "make contain". Cf. UL.UL[S], NIGIN[S]. (In AO 17 264, obv. 2-3 the sign indicates equality between shares, suggesting an identification with si_8, cf. íb-si_8).
C: 24, 65, n.3a.

/ lapātum, "inscribe".
T: 13, 20f, 32f, 105.44.
C: 29, n.13.

laqûm, "take".
T: 87, 105.25, 105.45.
C: 76.

/ libbi, "inside", from libbum, "heart", "bowels", "interior".
T: 13, 21, 31ff, 80, 105.25, 105.34.
C: 15, 67f.

lugúd~karûm, "short" / "shortened", cf. the commentary to gíd[S].
T: 105.25f.
C: 105.29.

/ lul, "false".
T: 105.35.
C: 35, 68, 71, 78, 105.35, 105.39, n.8, n.20.

maḫārum, "stand against" (a counterpart). Cf. meḫrum[S], mitḫartum[S], šutamḫurum[S]. Gt: "Stand against each other as counterparts" (cf. BM 13 901 rev. II,10,16).

/ makṣarum, "bundling". From kaṣarum[S].
T: 105.6.
C: 105.11f.

/ mala, "so much as".
T: 16f, 31, 33.

manātum, "share", "contribution". From manûm, "to count".
T: 87.
C: 87, 89.

maṣûm, "correspond to", "suffice to".
T: 105.24f.

maṭûm, "become small(er)".
T: 20.

/ meḫrum~DUḪ, "counterpart", from maḫārum[S], cf. mitḫartum[S].
T: 7, 11, 80, 93, 105.43, 105.45.
C: 7, 105.51, n.5, n.35.

/ mīnûm~ennam, "what".
T: 16, 31ff, 81, 84, 105.16f, 105.43f.

/ mitḫartum, "square figure", from maḫārum[S], Gt (mitḫurum), cf. meḫrum[S], šutamḫurum[S].
T: 12f, 20f, 49, 105.45 (in adjective form, "sides not equal").
C: 8, 24, 67, 105.53, n.9a.

/ muttarittum, "descendant", from waradum[S].
T: 105.15ff.
C: 105.15.

/ nadānum~sum, "give" (as a result).
T: 20f, 80, 84, 93, 105.17, 105.25, 105.34ff.
C: 105.25.

/ nadûm, "lay down" (e.g. in writing), "draw".
T: 11, 22, 80, 105.44f.
C: 22, 29.

napḫarum, "totality", "(sub)total" in accounting. From paḫārum.
T: 105.45.
C: 105.54.

/ nasāḫum ~ zi, "tear off".
T: 7, 11, 12f, 16f, 21, 32f, 49, 80f, 86f, 93, 105.6, 105.17, 105.45.
C: 64, 67, 105.36.

/ našûm ~íl (~nim[S]?), "raise", i.e. calculate by multiplication.
T: 13, 16f, 20f, 80, 84, 86ff, 93, 105.6, 105.16f, 105.25f, 105.43ff.
C: 15, 19-19a, 22f, 38, 61f, 64f, (73), 79, 82f, 105.19, 105.29, 105.32, (105.37f), 105.50.

ne, "that", cf. ba-a-šu ša[S].
T: 105.36.
C: 105.36.

/ nēpešum, "being-made". Used in the expression "such the being-made" which closes the description of a procedure. From epēšum[S].
T: 92, 105.44f.
C: 92.

/ NIGIN (~UL.UL[S]? ~lawûm, "surround"?). "Make surround", "surrounding" Cf. šutākulum[S], šutamḫurum[S].
T: 7, 92f.
C: 8, 12, 24, 64f, 94, n.3a, n.36.

/ nim (~elûm[S]? ~našûm[S]), "lift" ("go up" in Seleucid texts, i.e. count a difference arithmetically).
T: 3 (the Seleucid use), 105.37f.
C: 73f, 102 (the Seleucid use), 105.37f.

NU (~elûm?? "ascendant"??)
T: 105.35.
C: 105,39f.

paḫārum, "meet", "congregate". D: "gather", "collect", whence napḫarum^S.

/ paṭārum~du₈, "detach" (in the translations, I first used "find out" and next "undo"). Used subtractively for the detachment of a pre-existing part, and (mostly) for the finding of reciprocals.
T: 16, 20f, 32, 81, 84, 87, 93, 105.16, 105.25f, 105.35f, 105.43ff.
C: 64, 75, 105.36.

/ pirkum, "bar". From parākum, "place oneself transversely, "bar". On the relations to dal and tallum, cf. MCT, 48.
T: 105.16, 105.24ff, 105.34ff.
C: 105.16, 105.30.

/ pūtum ~ sag^S.

/ qabûm, "say". from which iqbu, "he said", the quotation marker.
T: 16, 80, 87, 92.

/ rēška likil, "that your head hold" (the translations used "that your head retain"). From kullum^S.
T: 80, 81, 105.16f, 105.44f.
C: 28, 79, 82, 105.50, n.5a.

/ sum~ nadānum^S.

/ sag~ pūtum, "width"
T: 16, 31ff, 86ff, 92f, 105.6, 105.24ff, 105.34ff, 105.43ff.
C: 35, 50, 97.

/ sag-dù~ santakkum, "triangle".
T: 105.6, [105.15].
C: 105.14.

/ sag-ki-gu₄ (~abusamikum^S?), "trapezoid" (at least normally presumed to be right, i.e. to possess one "length" which is the length).
T: 105.6, 105.34.
C: 52, 105.12ff.

saḫārum (~NIGIN), "to turn oneself".
Cf. asaḫḫir^S.
C: 24, n.3a.

sanāqum, "arrive", "come close".
T: 105.25f
C: 105.25, 105.33.

santakkum ~ sag-dù^S.

sapāḫum, "scatter", "dissolve".
T: 87 (?).
n. 30.

/ sar, a measure of area (= 1 GAR²).
C: 76, n.7.

/ ṣiliptum, "diagonal". From ṣalāpum, "cut through (diagonally)", "cross over".
T: 105.6.
C: 105.12.

/ šakānum~gar (~ U.UL^S?), "pose" (in some cases at least apparently by writing a number to a line in a figure).
T: 13, 16f, 20, 80, 84, 86f, 105.16f, 105.25, 105.35f, 105.43.
C: 23, 29, 48f, 64f, 79, 82f, 91, 98, n.9.

/ šakīltum, cf. takīltum^S.
C: 105.24, n.5a.

šâlum, "ask" (p. 92 used "search").
T: 92, 105.43, 105.45.
C: 75.

šapālum, "be low", "be small". Cf. šapiltum^S, cf. ki^S.

šapiltum, "remaining thing" after a subtraction etc. From šapālum^S.
T: 32f.
C: 31.

/ šiddum ~ uš^S.

/ šumma, "when".
T: 17, 105.24, 105.43f.

/ šumum, "name".
T: 92f.
C: 92.

/ šutākulum, "make span" (a rectangle or other quadrangle). Št-stem of akālum^S. So, the ~~literal meaning is "make eat~~ each other", whence probably "make two lines occupy a surface together, i.e. "determine a quadrangle with the two lines as (adjoining or opposite) sides". Cf. takīltum^S.

(the translations used above "give reciprocally" and "cross" should be abandoned"). Cf. kú^S.
T: 11, 13, 16f, 20f, 31ff, 49, 80, 92f, 105.16, 105.25f, 105.43, 105.45.
C: 11f, 16, 19, 19a, 37f, 62, 64f, 83, 105.19f, 105.24, 105.29, 105.31f, 105.33, 105.50.

/ šutamḫurum, "raise against each others (as counterparts)". Št-stem of maḫārum^S. Cf. meḫrum^S, mitḫartum^S.
C: 8, 12, 49, 64.

šutbum, "let leave", Š-stem of tebûm, "leave, set out, ..."; can be used as a complement to paṭārum^S, viz. as "detaching" 1/6 of something and removing it.
C: 64.

/ tab~ eṣēpum^S.

tabālum, "take away". subtraction of an already distinct entity.
C: 64.

/ takīltum, "(thing which) was made to span", cf. šutākulum^S. (The translations used, "thing to which was given" and "which was crossed" should be abandoned).
T: 11, 80, 105.16f, 105.45.
C: 105.16, 105.24, 105.52f, n.5a.

/ tamar, "you see". From amārum^S. An early Tell Harmal-text (see Baqir 1950) uses igi-dù in the same function. *(Should be tammar)*.
T: 7, 84, 87f, 92f, 105.43ff.
C: 29f, n.36.

târum, "turn back". Cf. atur^S.

/ tawirtum (~ íd?), "field part".
T: 105.15f, (105.34?).
C: 105.15.

temen, "field", "terrace",...??
C: 105.1, 105.4.

/ ugu ... dirig ~ eli ... watārum^S.

/ UL.GAR 1) ~kamārum^S. 2) ~kimrātum^S.
T: 92f.
C: 92, 98.

/ UL.UL (~NIGIN^S?), describes "construction" of rectangles and squares. Untranslated.
C: 49, 64f, 98.

UR.UR, C: 38, 64.

/ uš~ šiddum, "length". Can also be used in the generalized sense in which a quadrangle has four "lengths".
T: 16, 31ff, 86ff, 92f, 105.6, 105.24ff, 105.34ff, 105.43ff.
C: 50, 105.7ff, 107.28f, 105.43, 105.51, n.32.

Ú.UL (~šakānum^S?), a functional analogon of šakānum. Leave untranslated.
T: 92.
C: 98.

wabālum, "bring (to a place)"
T: 32.
C: 36.

warādum, "descend", cf. muttarittum^S.

/ waṣābum~ daḫ, "append". Cf. waṣūbum^S.
T: 7, 11, 13, 21, 31ff, 34, 49, 80, 87f, 92f, 105.16f, 105.26, 105.36, 105.43ff.
C: 24f, 64.

/ wāṣītum, "unit square"?? From waṣûm^S.
T: 13.
C: 78, n.7.

waṣūbum (~KI.GUB.GUB^S?), "extension". From waṣābum.
T: 93.
C: 93, 97.

waṣûm, "go out", "go away". Cf. waṣitum^S.

watārum ~ dirig, "go beyond (its measure)". Cf. eli ... watārum^S and la watar^S.

/ zi~ nasāḫum^S.

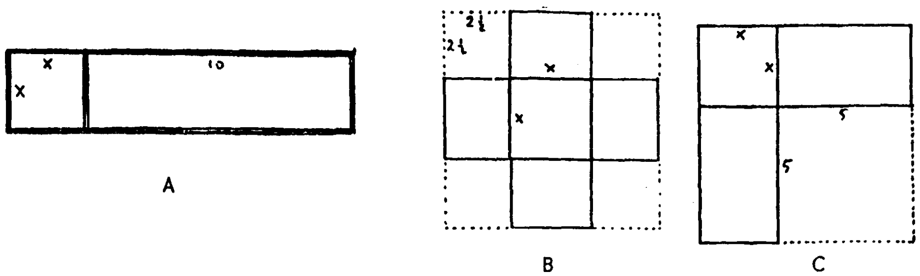


FIGURE 1

Al-Khwârizmî's justification of the solution $x = \sqrt{39+(10/2)^2} - 10/2$ to the equation $x^2+10x = 39$

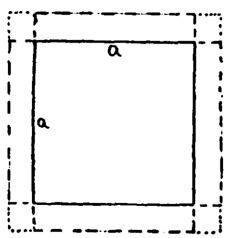


FIGURE 2

The approximation of $\sqrt{a^2+r}$ in IM 52301, according to E.M. Bruins.

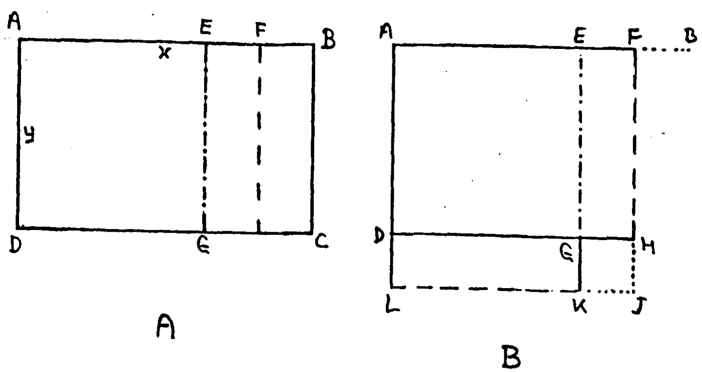


FIGURE 3

A geometrical reconstruction of TMS, Texte XIII, and of YBC 6967

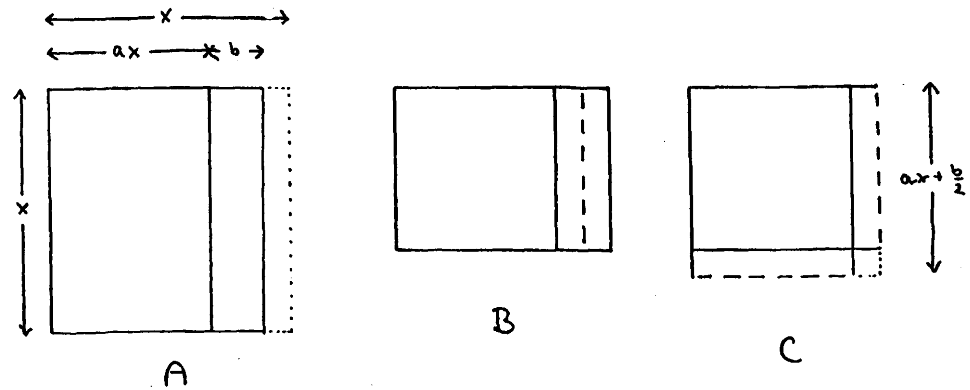


FIGURE 4

The geometrical reconstruction of the Babylonian solution to $ax^2+bc=c$, e.g. BM 13901 no. 3.

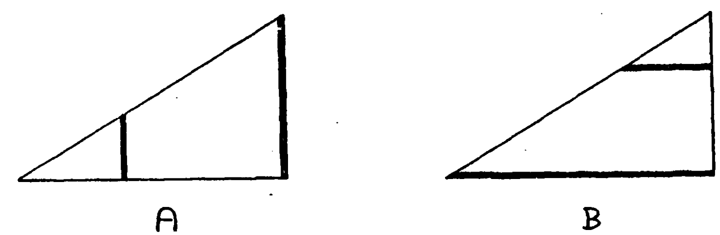


FIGURE 5

The geometrical analogy of the change of scale, expressed by the term našûm, "to raise", "to lift up".

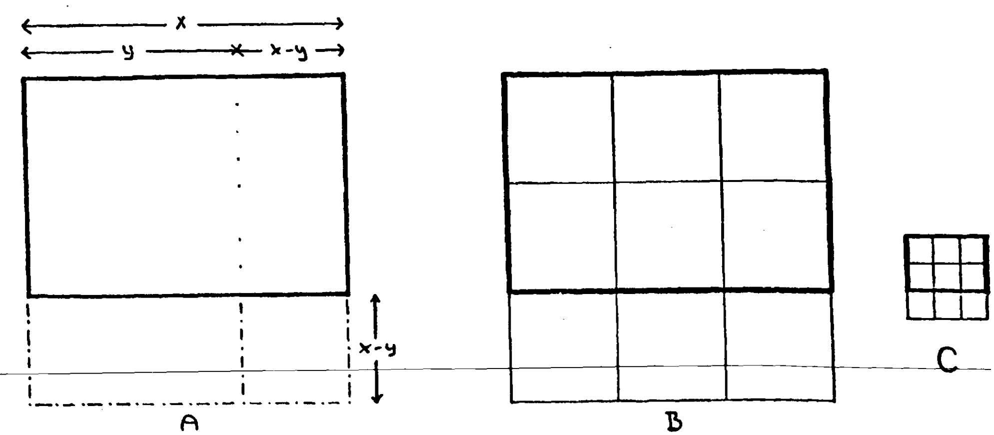


FIGURE 6

The geometrical representation of VAT 8390, problem 1. Problem 2, $y^2=4(x-y)^2$, can be followed on the same figure.

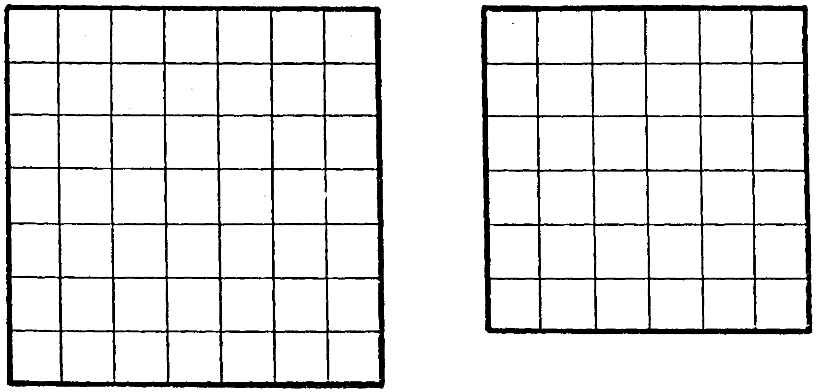


FIGURE 7

The two squares of BM 13901, problem 10, and their subdivisions.

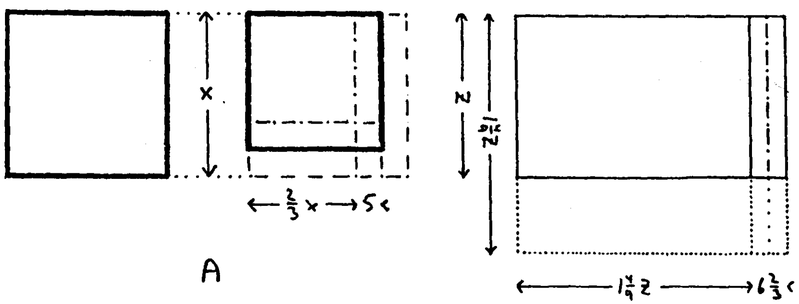


FIGURE 8

The two squares of BM 13901, problem 14 (8A), and the corresponding "squares and sides" (8B).

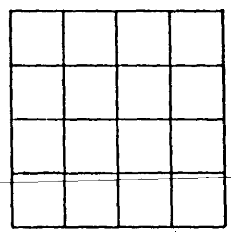


FIGURE 8̂

The subdivided square of BM 15285, problem 10.

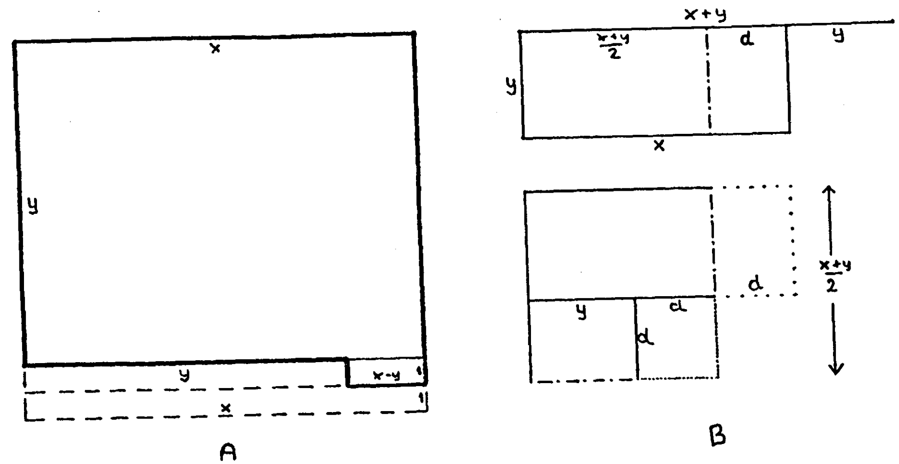


FIGURE 9

The geometrical reconstruction of AO 8862, problem 1.

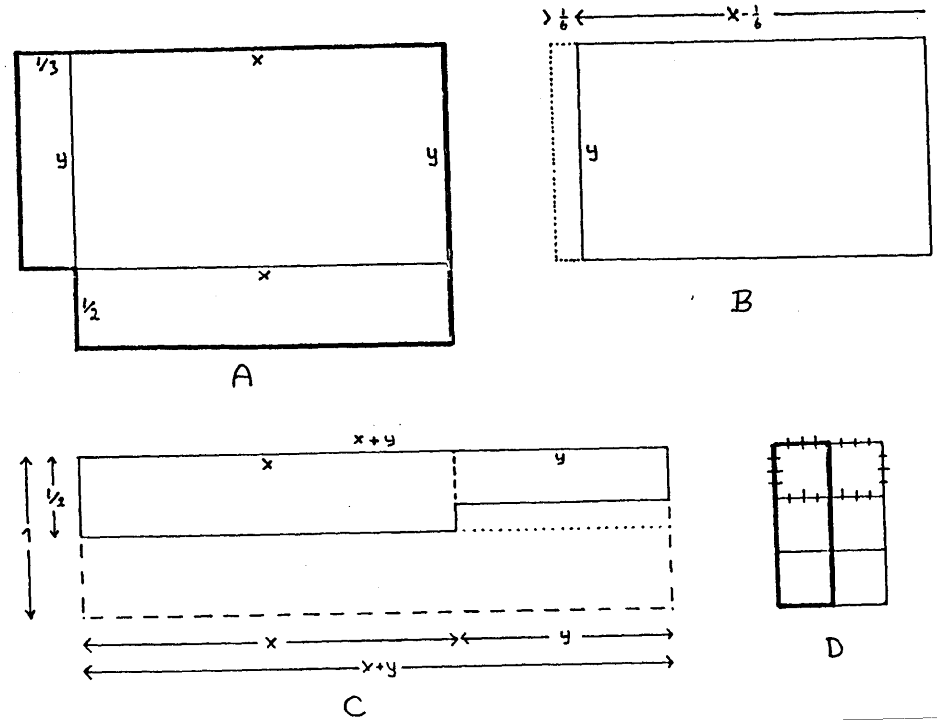


FIGURE 10

The geometrical reconstruction of AO 8862, problem 2.

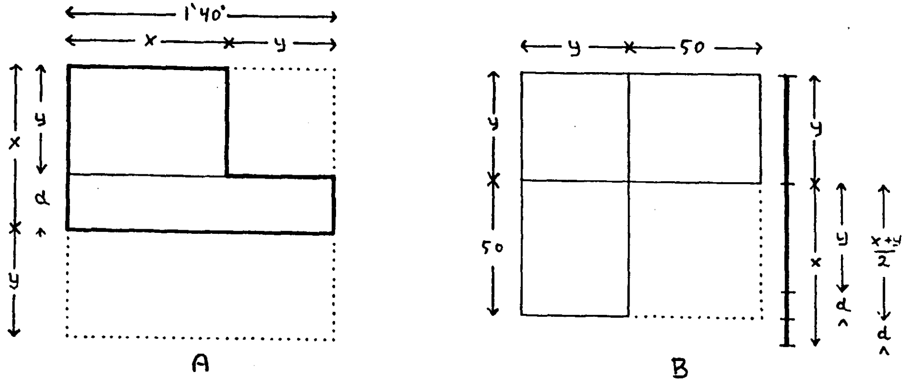


FIGURE 11

The geometrical reconstruction of AO 8862, problem 3

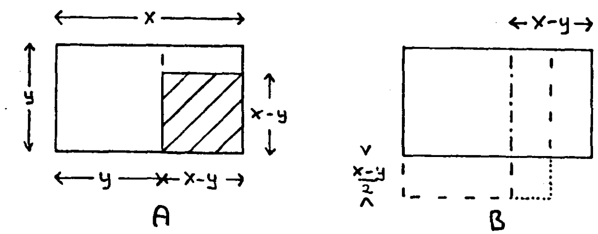


FIGURE 12

The geometrical reconstruction of YBC 6504, problem 1.

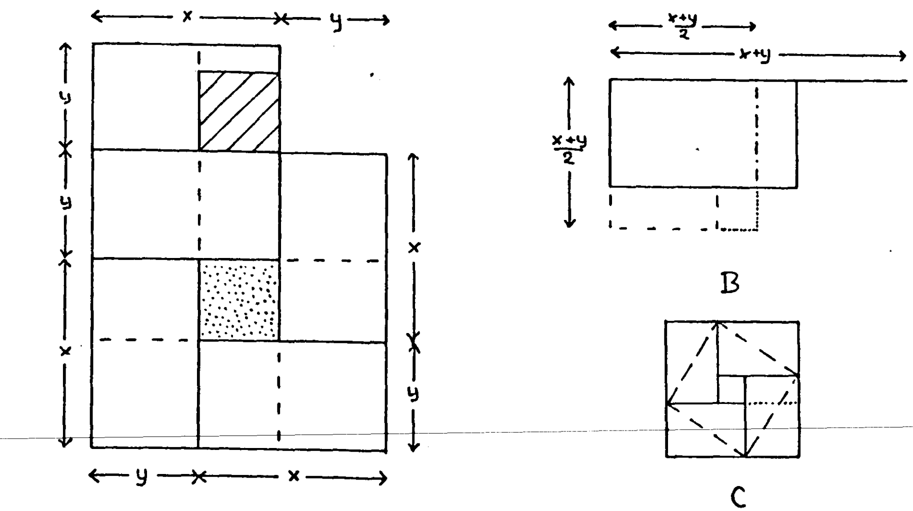


FIGURE 13

The geometrical reconstruction of YBC 6504, problem 2.

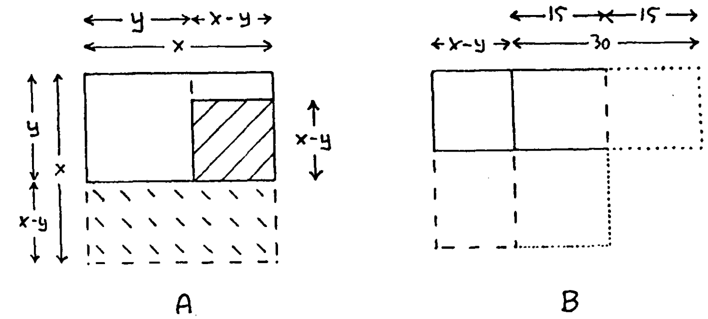


FIGURE 14

The geometrical reconstruction of YBC 6504, problem 3

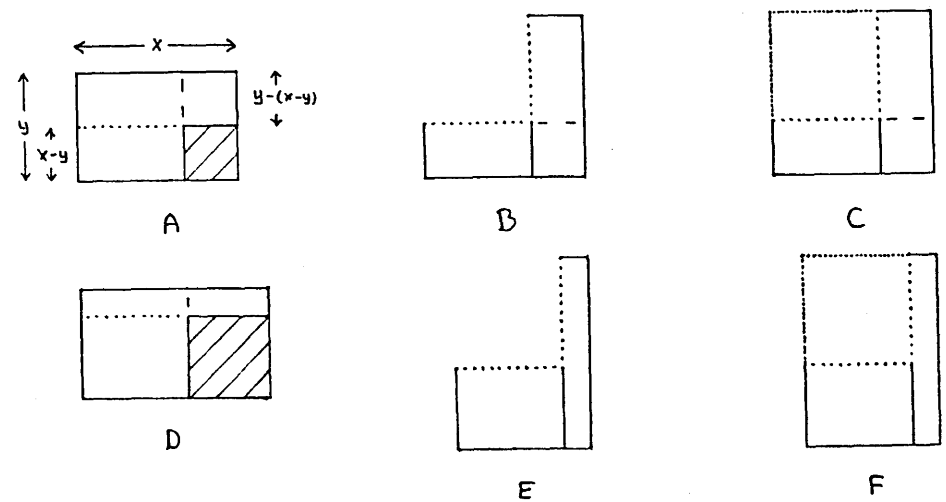


FIGURE 15

The hypothetical geometrical reconstruction of YBC 6504, problem 4, in correct (A-B-C) and distorted (D-E-F) proportions.

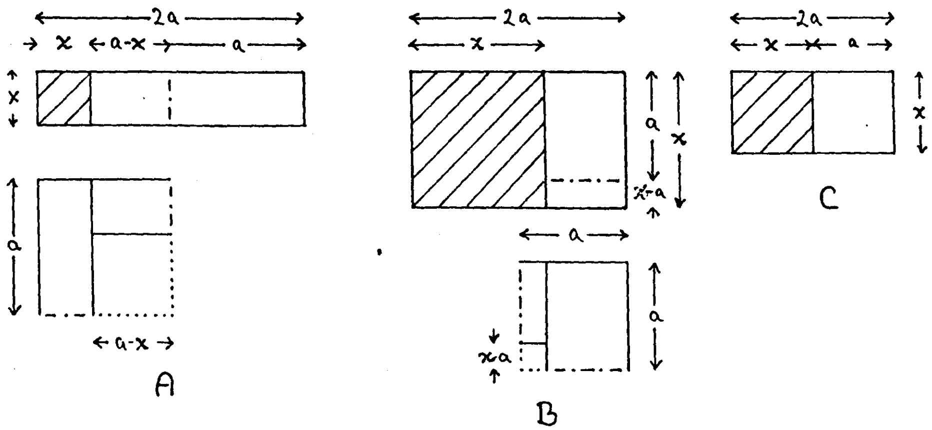


FIGURE 16

The graphical procedures for the solution of $2ax-x^2=b$, according to whether $x<a$ (16A), $x>a$ (16B) or $x=a$ (recognizable by $a^2=b$) (16C).

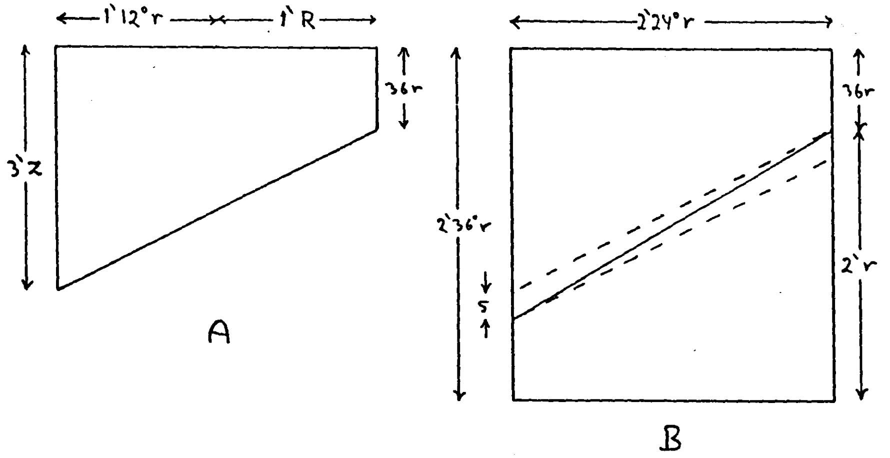


FIGURE 17

The trapezoid of VAT 7532 (17A) and the auxiliary rectangle used in the solution.

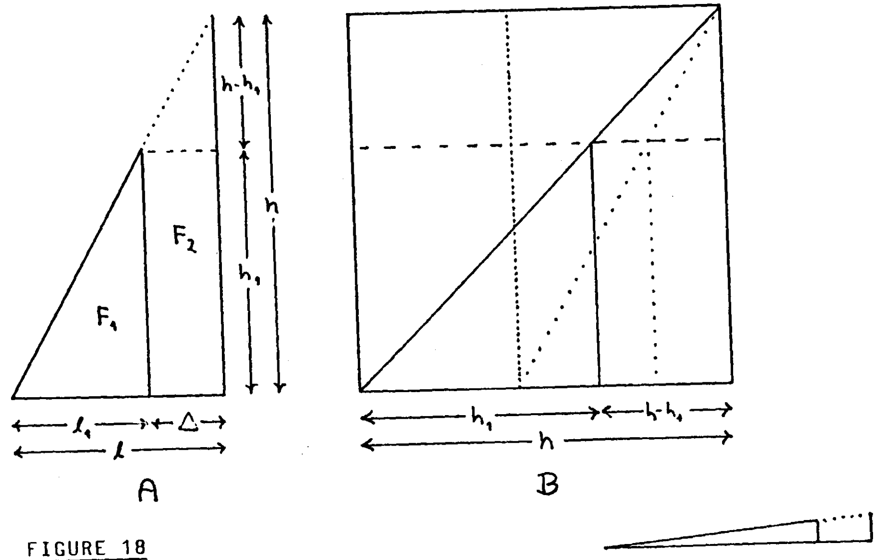


FIGURE 18

The siege-ramp of BM 85194 and BM 85210.
Neugebauer's terminology (MKT I, 183) is used.

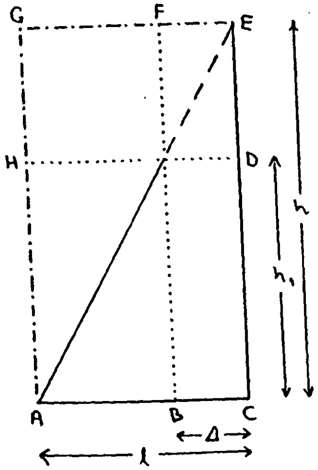


FIGURE 19

Kurt Vogel's reconstruction of BM 85194, the siege ramp.

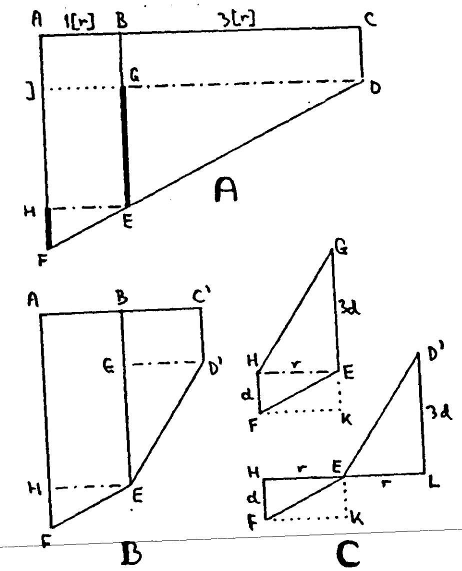


FIGURE 20

The geometrical reconstruction of Strassb. 367

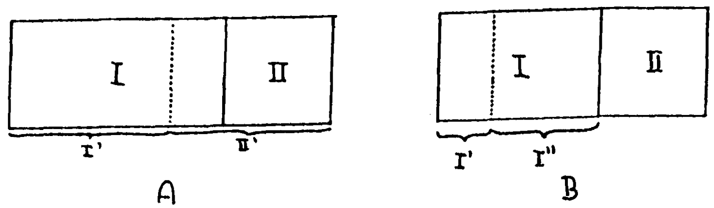


FIGURE 21

The two fields of VAT 8389 and VAT 8391.

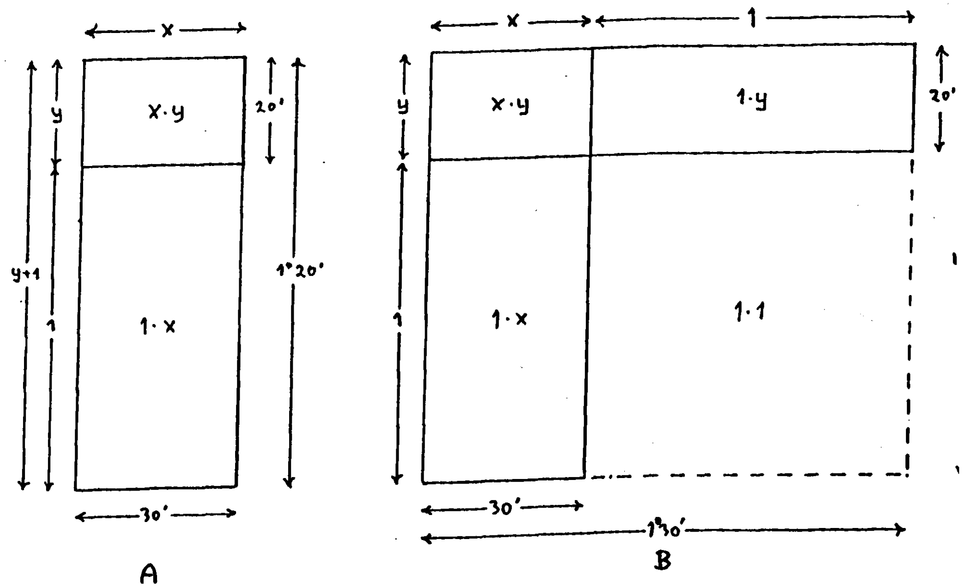


FIGURE 22

The basic transformations of Susa-text IX.

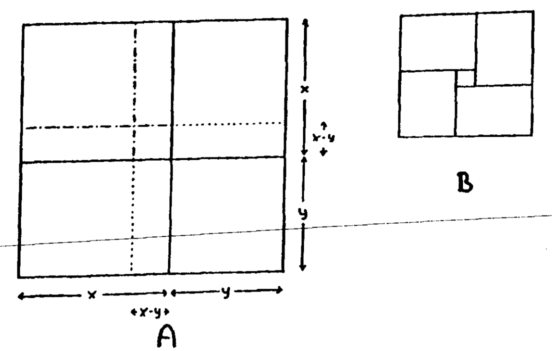


FIGURE 23

The eventual geometrical reconstruction of BM 34568, problem 9.

NOTES

1. In this case, my translation is based on Neugebauer's translation in MKT III, but
g had to the transliteration.
All subsequent translations I base on the transliterations, but of course with an eye to the translation given in the source quoted.

The sexagesimal numbers are rendered as by Thureau-Dangin (in TMB) and Bruins (in TMS): n`` means $n \cdot 60^2$, n` means $n \cdot 60$, n° (or n) means n, n′ means $n \cdot 60^{-1}$, n″ means $n \cdot 60^{-2}$, etc. In a translation which tries to be as close as possible to the phrasing of the original text, this system must be preferred to the sexagesimal place value notation introduced by Neugebauer, because it avoids the introduction of most of those zeros never present in the original text, but still permits the identification of "sexagesimal levels".

It should be kept clear that a notation of the order of sexagesimal order is only anachronistic with regard to the written text. The Babylonian mathematical texts contain so few mistakes of order of magnitude in additions and subtractions that their authors must have been in possession of some means to keep track of correct orders of magnitude. The texts VAT 7532 and VAT 7535 (MKT I, 294f and 303ff shows us at least one such means: Speaking not of 3 when 3` is meant but of 3 šu-ši, "3 sixties".

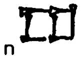
Numbers written in the original text by means of words will be rendered by words.

2. [Deleted]

3. Normal-type transcripted words stand for undeclinable word-signs (ideogrammes), of which the Sumerian value is given, and for Sumerian expressions; versals render the current names for cuneiform signs (and thus may or may not correspond to the phonetic value of the sign); underlining stands for Akkadian phonetic writing. In all essential places, the translation tries to render the grammatical form, e.g. the verbal tense and mode, as precisely as possible (here, as in the following, I used GAkGr).

3a). The Akkadian translations of ideogrammes given by Bruins and Rutten are often free translations which render a word-sign not by its real Akkadian equivalent but rather with a term which according to the authors could be expected in this mathematical context. Such retranslations from French into Akkadian may perhaps be advantageous from some point of view, but in the context of the present investigation they are useless. So, the equivalents offered in TMS for **NIGIN**, daḫ and íb-si in the translation of Texte XIII have to be disregarded.
In the text in question, **NIGIN** is only used for squaring

but in Texte XXI, line 5 it is used for the multiplication of 5 with 2, and in IX.5 correspondingly. Since the ideogram stands for such words as lawûm, "to surround [a garden, a field, etc.]", saḫarum, "to turn oneself" (also with connotations of "periphery"), the translation "to turn" is chosen, viz. as the sides of a rectangle are turned so as to constitute a frame.

The sign itself is suggestive:  (ABZ p. 31, no. 529, Old Babylonian form). Half of it, the sign LAGAB , is used in AO 17264 (MKT I, 126; Thureau-Dangin 1934a:61ff) as an ideogram for equality, probably because of the recurrent association between the square figure and the concept of equality (cf. ib-si₈ and šutamḫurum). The geometrical suggestiveness of the term and the association with the equality-concept will have lead Thureau-Dangin to assume the equivalence NIGIN~šutamḫurum at a time when no instances of multiplication of unequals were known to be described by the term (but still a text where the term denotes half the periphery of a rectangle, BM 85200+VAT 6599, Obv.II.11,23 - MKT I,197). In the light of the evidence provided by TMS, Texte IX and XXI. Thureau-Dangin's hypothesis now seems untenable.

A better choice would probably be "make surround"; even the composite ŠU.NIGIN used for sums in accounting can be interpreted as that which results when your hand (šu) surrounds everything, collecting it into one place (Killian Butz, private communication). The other use of the term, half-periphery, could then be translated as "surrounding", i.e. "that which surrounds" in the sense of Euclid's Elements II, def. i: "Any rectangular parallellogram is said to be contained by the two straight lines containing the right angle".

As LAGAB is known to have the same ideographic value lawûm, the same translation might be used. Alternatively, the probable origin of the sign as a picture of a container (ŠL II,iv, N° 483) might be taken as a suggestion to choose "make contain". (In the rather late AO 17 164 and the equally late Susa texts, it is found where si₈ or íb-si₈ might be used, suggesting distinction from NIGIN).

4. Bruins and Rutten (TMS p. 83) understand 1(kam) and 2(kam) as "firstly" and "secondly" instead of "the first" and "the second". Several considerations speak against this interpretation. FOr one thing, the normal function of these forms is that of ordinal numbers, while in certain cases they are used where we would expect cardinal numbers (one instance in VAT 8528, Obv. I.23f, MKT I, 353). Moreover, they stand inside the phrases of addition and subtraction, where only identifications of the terms involved could be expected (many instances of such constructions follow below).

5. "Its equal" is the customary translation of the expression meḫeršu. At least in one mathematical text, however, it can only mean "its opposite", viz. the opposing side to one of the unequal parallel sides in a trapezoid (see Baqir 1950a:132). In another, earlier Tell Harmal-text, DUḪ, (ABZ p.102,no.167, and MEA p.107) which may be an ideogram for "its opposite", is seemingly used in a related sense (see Baqir 1950:42 1.10, and commentary p. 53), viz. as a line forming an angle with the other line which it "opposes". The choice of the translation "opposite" would support a geometrical interpretation of the text. In order

A translation reflecting both senses of the word would be "counterpart" - cf. below, pp. 105.43ff and 105.51.

Cf. addendum p. 7.

not to defend the geometrical heuristics by possibly shaky arguments I have preferred the customary translation; still, the connotation of equal, different sides need not be forgotten.

5a. Neugebauer & Sachs (MCT p. 130) mention a proposal due to Thureau-Dangin, that takīltum designate "a number which has been multiplied by itself" and point to a text where this interpretation makes no sense (incidentally the text during the discussion of which Thureau-Dangin proposes the interpretation quoted as a related meaning - 1937a:23). However, takīltum is in any case related to šutākil (discussed below), "to make receive one another" or "to make hold one another", but an abstract nominal, non-reflective form meaning "that which was made to receive" or "... made to hold" (cf. GAkGr §56[1]). This fits even the counterexample of MCT, better than Thureau-Dangin's "coefficient".

6. Ištēn, "one", cannot be an ordinal (cf. Goetze 1946:197). So, ištēn ... ištēn, literally "one ... one", is safely translated (as in MCT) as "one ... the other".

Addendum: *The question of the takīltum turns out to be more intricate than I had originally assumed. In his review of TMS, von Soden (1964:49) pointed out that text XIX, problem D, line 12 refers to "6 40 the takīltum that your head retains". This number 6 40 has not been given reciprocally. On the other hand, all other Susa occurrences of the term refer to numbers which are not retained: In the same problem, line 10, 3 20 (found as half the 6 40) is a number which has been given reciprocally for a quadratic completion; in XXIV, line 18 and (with a small variation) XII, line 14, the same function is fulfilled; finally, the occurrence in XXI,1.16 is isolated by a break in the tablet and therefore not to be dechiffered (but anyhow nothing is retained in the foregoing text). So, the identification in XIX D, line 12 would, if derived from kullum, be quite exceptional. Instead the idea could be that the two halfs into which 6 40 is broken span a square as they are given reciprocally, and that therefore even 6 40 does so when bent in angle. This would fit well with the reference in line 10 to "3 20 the 2 takīltum", which is again partly deleted, probably because only one of the two "identical copies" of 3 20 is operated upon.*

In VAT 8512, obv. 19, a number 21 is referred to as šakīltum, while in obv. 12 it was "retained" and in obv. 15 referred to as "21 which your head retains". However, in obv. 15 it is also given reciprocally, and in obv. 23 it is referred to as "given reciprocally". Close analysis of the structure of the text points to parallellism between obv. 19 and obv. 23, and therefore to an interpretation as "the thing that was given reciprocally" even for šakīltum. Cf. below, p. 105.16.

So, neither of the texts which might point to a derivation of takīltum or šakīltum from kullum appear to do so when closely inspected. On the other hand, KÚ, the ideogram for akālum and besides that for šutākulum, can also stand for takīltum (according to MEA p. 55, no. 36). Besides, occurrences like that of YBC 6967 (of which there are many) are of a type that cannot possibly refer to "the thing retained by your head" since no head is ever referred to in the solution of simple mixed quadratic equations.

7. [Reformulated, December 1984].

In the traditional arithmetico-algebraic interpretation, the term wāṣītum has always been somewhat enigmatic. With all reservations, Thureau-Dangin (1936a:31) understands it as a verbal specification of the 1 as one, not 60^n, $n \neq 0$ (cf. also TMB, 1-3, passim). Neugebauer (MKT III,11) rejects this for the reason that such a specification would be just as necessary in many other instances than second-degree-equations with one variable, but there it is never found; instead he proposes, equally with all reserves, the translation "coefficient". Still, even this interpretation is problematic, since the problem as stated does not contain any coefficient 1. Still, analysis of the occurrences of the term in the present tablet shows that the 'coefficients' of first-degree-terms of the type αx are calculated as "α times 1, the wāṣītum". (In the present problem, this becomes unclear because the scribe confuses the two numbers "the third", cf. also line 12).

In my first edition, I could get no closer to a solution. At closer inspection, however, the term turns out to be an important clue. It derives from waṣûm, "to go away", "to go out", "to project" "to stick out". In non-mathematical contexts, the term itself may, inter alia, designate something projecting from a building (or the sprout projecting from the date-palm).

This turns out to be meaningful in our geometrical interpretation. In fact, in second-degree-equations in two variables (e.g. YBC 6967, discussed above, p. 11), the statement of the problem leads to no difficulties of dimension; in problems of one unknown, however, where a surface and a number of "sides" are added, the latter have to be considered as areas - i.e., a "side" x must be understood as a rectangle of length x and width 1. This width is, precisely, projecting from the side.

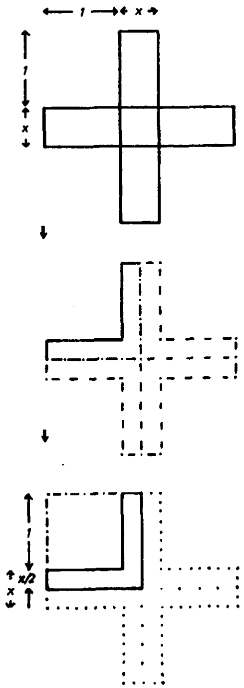
Firstly, this explains why the wāṣītum is absent from problems of two unknowns. Secondly it gives a meaning to the curious multiplication ("raising") of the coefficient α by a "number" 1 - by this process, the rectangle of width 1 is transformed into that of width α, i.e. into that rectangle which is to be used in the geometric procedure.

This interpretation of the term can be tested against one of the most enigmatic problems of the tablet, viz. Nº 23 (MKT III, 4f; cf. TMB, 17f). It can be translated as follows:

Reverse II

11. The surface of the four fronts and the surface I have accumulated: 41′40″.
12. 4, the four fronts, you inscribe. The reciprocal of 4 is 15′.
13. 15′ to 41′40″ you raise: 10′25″ you inscribe.
14. 1 the wāṣītum you append: 1°10′25″, 1°5′ the side.
15. 1 the wāṣītum which you have appended you tear off. 5′ to two
16. you double: 10′ nindan is the square.

At first, this calls for some philological comments: "Front" translates pūtum, which is often considered the Akkadian equivalent of s a g, "width". However, in no normal "length-width"-text is s a g replaced by this Akkadian equivalent (the two occurrences, YBC 9897, in MCT, 90, and IM 53965, in Baqir 1951:39, are not normal). So, the Akkadian word must have been used by special intent, as different from "width". The construction "the four fronts" renders an Akkadian construction pāt erbettum which appears to reflect that four fronts are spoken of which invariably belong together (like in the analogous expression "the seven mountains", cf. GAG § 139i). The translation of line 11 is chosen instead of that of Neugebauer and Thureau-Dangin ("A surface; the four fronts and the surface I have accumulated") because both "surfaces" are pointed out explicitly (by a grammatical complement lam) to be accusatives, a fact which is neglected in MKT as well as TMB. In line 16, "is the square" ought in reality to be translated "stands against its equal" (cf. above, p. 8, on šutamḫurum and mitḫartum).



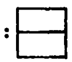
The procedure of the problem is shown to the left. Truly, the first step (the division by 4) is only stated as an arithmetical operation in the text(lines 12-13). In line 14, however, the "1" which is added is spoken of as the wāṣītum itself, not as its square (and line 15 shows that this is intended and not an elliptic confusion). This coincides with the Old Babylonian habit to identifie a square (considered as a geometric configuration) with is side (a habit which is discussed below, p. 24). So, once the geometric meaning of the wāṣītum is clear, the whole procedure becomes transparent; without that it is so unexpected that Neugebauer suggested the possibility that the problem is a confusion of something else which happens to make mathematical sense (see MKT III, 14).

Apart from its occurrences in a number of problems in BM 13901, the term is found in the same function in AO 6770, N° 5 (see MKT III, 65, and TMB, 73). Besides that, the masculine form wāṣûm is found in two other texts in a somewhat different function, which may however be elucidated by the above.

One of these occurrences is in VAT 8391 N° 3 (Rev. I,3-II,9; MKT I, 328f). The problem deals with two fields, of which the

first yields a rent of 40′ qa per s a r and the second 30′ qa per s a r. The areas are known to differ by 10` s a r, and the total rent is 18`20° qa. Among the things to be "posed" is "1 the wāṣûm", which was not mentioned in the statement of the problem, and which must therefore be something known in advance, a standard-concept.

Firstly, the text calculates the rent of that part of field I by which this field "goes beyond" the second field: 6`40° qa. This is subtracted from the total rent, and a rent of 11`40° qa remains which must be due to equal areas from the two fields.

Now "1, the wāṣûm" is broken into two halves, and what is calculated in the next step turns out to be the rent (35′ qa) of a unit area (1 s a r) composed from equal portions from each field; from this rent of an "average s a r" the remaining total area is found by division. So, functionally, "1 the wāṣûm is nothing but a unit area. What may be meant by the term may, however, be different: that width of 1 which transforms the basic unit of length (the GAR) into the basic area unit; if this is so, the bisection of the average sar is though of in utter concreteness, as a geometric bisection of the square:  .

The other occurrence of "1 the wāṣûm" is in VAT 8528 N° 1 (MKT I, 353). This is a problem of compound interest, the procedure of which is, on the whole, rather obscure. I shall not discuss it here but just mention that a number "32" is possibly visualized in the following way: , where the wāṣûm is the length 1 projecting to the left from the rectangle 15·2.

(Neither of these interpretations, of VAT 8391 N° 3 and VAT 8528 N° 1, have to my knowledge been proposed before).

8. [Deleted].

9. The apparent semantic neutrality of šakānum may perhaps be a lure. In mathematical texts, it occurs in places where it is often interchangeable with lapātum ("to touch", "to take hold of", "to inscribe"), and with nadûm ("to lay down", "to draw") (for references, see the glossaries of MKT and TMB; cf. also Thureau-Dangin 1937:88). In administrative contexts, the semantic span of the term includes "to put down in writing", "to submit an [oral or written] report" (BAG p. 269b). So, maybe the term had graphical connotations in the mathematical context, related e.g. to the numbers written along the dimensions of "fields" in many texts

9a. The "figure the sides of which are equal" is also Thureau-Dangin's proposal for the meaning of the corresponding term i b - s i (1934:51).

10. [Deleted].

11. The addition is expressed by the term gar-gar-ma, where the particle "ma" expresses a conclusion (":"). From parallel passages in the text (e.g. Obv.I.37) and from a reference to the sum as kimrātum it is clear that gar-gar must be read ideographically for kamārum "to take together". Gar-gar is no new operation.

Cf. p. 98, marginal note)

12. The phrase "ba-a-šu ša X" occurs three times in the text: I.12, II.19 and III.13. By the first two occurrences, X is a number which has just been calculated, and so the translation "half of it, of X" would seem adequate; however, in the third case, X is no immediately present number, and so the demonstrative interpretation "that" of the particle šu enforces itself. Since the construction occurs at mathematically analogous points of

Cf. p. 105.36, the marginal note, on a related use of the deictic particle ne.

the calculation (see discussion below), and since the demonstrative interpretation is possible in all three cases, the same translation is used in all cases.

13. Neugebauer's translation of the word nalpattum (from N-stem of lapātum) as "inverted" (MKT I, 114) seems unfounded. I follow Thureau-Dangin's interpretation (TMB p. 66), cf. GAkGr §56[c].

14. This geometrical interpretation of the transformation performed was first proposed by Kurt Vogel (1933:79). Independently, James K. Bidwell has pointed out the existence of a natural geometrical mapping of the calculations performed in the text (personal correspondance).

15. Neugebauer's proposal (MKT I,113 n. 11), that the expression is meant as a general term for something resulting from a calculation, is hardly acceptable as long as only quantities are so labelled which in the geometrical interpretation are in fact surfaces. A generalized meaning as "the value of a second-degree polynomium in one or more variables", which is the only non-geometric explanation of the term true to its range of application, I find much too abstract to be expected by the Babylonians. Furthermore, had such a concept existed, one should rather expect a term derived from one of the concepts of multiplication

than a term designating originally "a field" or its surface.

16. In this respect, of course, it is no different from symbolic algebra. We would also choose freely, after numerical commodity or according to our fancies, which variables to eliminate. The contrast is not with modern mathematics but with an image of Babylonian mathematicians according to which the search for standard-types determined everything.

17. Nothing is explained by the observation that šu ša, "that which is", is more easily written than ki-im-ra-at uš ù sag, "the accumulation of length and width". Still easier would be the writing of "half of 1‵40'", and if only the "accumulation" was thought of, the latter expression would tell just as much as the one really used.

18. I have corrected three errors in obv. II, 30-31: 1‵‵‵57‵‵21‵40' for 1‵‵‵57‵‵46‵40' (twice) and 17‵‵21‵40' for 17‵‵46‵40' (cf. Neugebauer's notes to the transliteration). When the square-root of the latter number is extracted, the correct value 4‵10' turns up, which shows that the scribe knew the correct result in advance. Two explanations of this (apart mutual cancellation of errors) seem possible: Either the scribe was copying a correct original while following the calculations in his head, thereby transmitting one initial error through several members of the text; or he constructed himself the problem from known numbers, in which case he could know what the square-root had to be.

19. The same interpretation was already given by Vogel (1959:51). It was proposed again by Goetsch (1968:102). I repeat it here because of its importance for the investigation of the relation between geometric and non-geometric arguments in Babylonian algebra.

20. That an auxiliary rectangle is considered is told directly in the parallel texts, which speak of a "lul" length, a "lul" width and a "lul" surface calculated as the product of the length and width. The term lul (GIR in MKT I, cf. however MKT III, 58, and Thureau-Dangin 1936b:161), literally "false", is the complement of the "true" [width] which we met in AO 8862, problem 1 (p. 32) which was to be distinguished from the width of an auxiliary figure used during the calculation.

21. Besides the examples discussed in the above, I can't abstain from mentioning the wonderful geometrical explanation of an intricate partition of a triangle in VAT 8512 first proposed by Solomon Gandz (see Gandz 1948:36f, or Vogel 1959:72)

Cf. the appendix, pp. 105.15 -105.23, where the text is translated and discussed.

22. For reasons of space and time, I shall go no further into the discussion of such limiting cases. However, I shall advance the guess that if the Babylonians were in possession of a general justification of their formula for the bisecting transversal of a trapezoid (i.e., if they did not extrapolate from single or simpler cases, as they extrapolated

Cf. the appendix on VAT 8512, pp. 105.20-105.22.

from trapezoids to other quadrangles), it must have been built on a geometrical heuristic involving some general scaling.

23. The use of the term in more generalized senses (the linear extension of the cube ≈ the cube root, and possibly the root of an exponential equation) fall outside the scope of this paper. Cf. MKT II, 11-35, "Glossar". and TMB pp. 215-243, "Lexique".

24. Of course, these generalizations build on a much greater material than the few texts discussed in detail above. A key to other occurrences is provided by the glossaries of MKT, TMB, MCT and TMS.

25. So, we have another case of several terms with closely related non-technical meanings designating the same operation (cf. p. 67), an indication that this non-technical meaning cannot be considered forgotten or irrelevant.

26. For this final part of the calculation, already Neugebauer (MKT I, 263) preferred a geometrical explanation.

~~27. This same unsophisticated way of reasoning~~ was proposed (for the case "$Y_I - Y_{II}$ given") by van der Waerden (1975:67) and by Kurt Vogel (1960:89f).

28. Parts of the text are heavily damaged, but almost all of it can be reconstructed with great certainty from parallel passages in problem 2. I mostly followed Neugebauer's transliteration; however, in line 4 (which has no parallel in problem 2) I followed Thureau-Dangin's reconstruction (TMB p. 115), which agrees with corresponding passages of other texts.

29. This is important for the interpretation of AO 8862, problem 2, where the geometrical interpretation of the calculation of $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ is confirmed (see p. 37 and Figure 10A).

30. I cannot make the transliteration ta-wi-iḫ (TMS p. 92) agree with the autography of the last sign. In any case, the problematic sign AḪ (ABZ no. 398) had no proper existence in the Old Babylonian period.

Von Soden (private communication, cf. 1964:49) proposes sapaḫum, "to scatter". Indeed, 15 is "scattered", i.e. split up into 10 and 5.

31. This doesn't give much sense, except that, in the transformed equation, 4x-4y+y=1`. However, I can propose no better reading; the sign read 4 may instead be read GAR, but that seems to be absolute nonsense.

32. The emphasis of the one length, etc., may remind of the habit to pose 1 for an unknown quantity which we have encountered a few times (cf. one instance p. 52, VAT 7532; further references in Thureau-Dangin 1938a). Indeed,

we may have stumbled upon the decisive elucidation of this habit: The calculation is not made under the assumption that the unknown quantity be the number 1; instead of unity, the unknown quantity is taken to be a "unit of accounting", in terms of which it is of course "1". (In any case, from a modern point of view this is of course the real mathematical contents of the habit).

33. The transcription gives 1, but since 1˙20′ is clearly to read on the autography, this seems a misprint.

34. Restitutions of the text where an Akkadian phrase can be reconstructed with fair certainty (from parts of words, immediate parallel constructions etc.) are put into [ ]. Restitutions of the meaning where the exact phrase can only be guessed are put into [( )]. Minor, fully certain reconstructions are not indicated (they will be found in TMS).

Cf. addendum p. 7.

35. DUḪ can be an ideogram for mihrat, "opposite", from maḫārum; in this context, it is natural to think of that other derivation of the word which often occurs in a similar place, "A into two break, $\frac{A}{2}$ and $\frac{A}{2}$ its equal/opposite (meḫeršu, see note 5) pose". Cf. ABZ p. 102, no. 167 "DUḪ".

TMS transcribes by a derivation of paṭarum, ~~"to loosen", "to detach", which is possible but~~ would be meaningless in the context - a difficulty which the translation tries to get around by changing "detach" into "break" - paṭārum into hipûm

36. All occurrences of the expression "you see" (tamar) in MKT and MCT belong to Goetze's group 6 ("northern modernizations of southern originals" - MCT p. 151), exception made of one later text and of those which cannot be ascribed to any group by linguistic criteria. Likewise, all appearances of NIGIN as a term for multiplication belong to the same group.

In the equally northern, Late Old Babylonian Tell Harmal texts, the term tamar is also found (see Baqir 1950a and 1951).

37. See Powell's selection of such texts (1976) and Vogel's paper (1958) on the continuity from Sumerian to Akkadian mathematics. Cf. also the discussion of the distinctive features of Sumerian and Akkadian mathematics and of the possibility of lacunae in the Sumerian material in Høyrup 1980:20ff.

38. So, Old Babylonian algebra constitutes another instance of a relationship recently treated for other phenomena from the history of scientific thought by Peter Damerow and Wolfgang Lefèvre et al (1981).

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[Since my knowledge of Russian is utterly scarce, I have only been able to cheque a few concrete questions in this book. It is possible that Vajman has already made several of the points which I discuss in this paper (although, as far as I have found out, the main evidence for the presence of geometrical methods in Babylonian algebra is not presented in the book). In that case I apologize not to have been able to recognize his results].

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